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Transient Solutions to a Stochastic Model of Ventilation

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By considering initial conditions and input parameters as random variables, a two cell conceptual stochastic ventilation system model is developed and the resulting stochastic differential equations solved. Based on species conservation equations, the model is capable of determining the concentration-time trajectories for a specified confidence interval.

Two sample cases illustrate the flexibility of the model in characterizing diverse ventilation systems, and in determining the impact of input parameter uncertainty or randomness on the time behaviour of concentrations.

NOMENCLATURE

- A_1 portion of Cell 1 surface area across which flows occur
- A_2 portion of Cell 2 surface area across which flows occur
- Cov covariance operator
- $C_t^{*}*$ vector stochastic process
- $C_1(t)$ instantaneous Cell 1 contaminant concentration
- $C_2(t)$ instantaneous Cell 2 contaminant concentration
- $C_\infty(t)$ instantaneous ambient or surroundings contaminant concentration
- $C_1^*(t^*)$ instantaneous Cell 1 non-dimensional contaminant concentration
- $C_2^*(t^*)$ instantaneous Cell 2 non-dimensional contaminant concentration
- $C_\infty^*(t^*)$ instantaneous ambient non-dimensional contaminant concentration
- $F(C_t^{*}*)$ drift component coefficient matrix
- $G(C_t^{*}*)$ diffusion component coefficient matrix
- K_i grouped coefficient term, a random variable of constant uncertainty
- $\bar{K}_i$ mean component of coefficient term
- K_i' stochastic component of coefficient term
- LEV local exhaust ventilation
- $\dot{m}$ contaminant source strength, kg mol s^{-1}
- Q_e mean flow exhausted from exhaust hood
- Q_f mean flow across exhaust hood face
- Q_i mean flow between Cell 2 and workplace surroundings
- Q_m mean makeup air flow supplied as general dilution ventilation
- $\tilde{Q}_f$ turbulent component of exhaust hood face flow
- $\tilde{Q}_i$ turbulent flow component between Cell 2 and workplace surroundings
- RMS root mean square amplitude
- SDE stochastic differential equation
- t^* non-dimensional time, $t^* = t/\tau$
- Var variance operator
- v_f' turbulent velocity component of flow across exhaust hood face
- v_i' turbulent velocity component of flow across workplace exterior
- $\forall_1$ Cell 1 volume
- $\forall_2$ Cell 2 volume
- W_t Weiner or Brownian motion process
- dW_t independent and stationary increment of Wiener process
- γ_{ij}^2 correlation structure of coefficient K_i with K_j
- κ cell volume ratio, $\kappa = \forall_2/\forall_2$
- ω probability state variable
- ϕ fraction of contaminant release rate confined to Cell 1
- σ_i^2 correlation structure (variance) of coefficient K_i
- ξ_t stationary Gaussian white noise process

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1. INTRODUCTION

WITHIN THE engineering disciplines, most dynamical models of physical systems have been carried out in a deterministic framework. That is, given sufficient information at one instant in time the deterministic model "exactly" determines the entire future behaviour of the system. In contrast, probabilistic models assume that regardless of how much is known about a system at a given instant, it is impossible to determine with absolute certainty the future behaviour of the system. Probabilistic models, then, are mathematical models which include uncertainty and randomness. Some recent examples of probabilistic models of various physical systems may be found in [1–3].

A probabilistic approach is adopted here by considering that many of the phenomena influencing a ventilation system are inherently random, or at the very least, quantifiable only in a statistical sense, thus implying a measure of uncertainty. Through use of the tools of stochastic analysis, a conceptual two-zone stochastic model of a ventilation system is developed. The resulting solutions provide concentration values over time as a stochastic random variable with confidence interval ranges.

This paper focuses on transient solutions to the model. Two sample cases are examined to illustrate the model's usefulness. This work is an extension of previous research on steady-state solutions to the model [4]. For completeness and for ease of understanding, some of the theoretical treatment given in [4] is repeated here.

The reader who is unfamiliar with the concepts of stochastic processes, stochastic differential equations [SDE's], and stochastic calculus is referred to [5–7].

2. DEVELOPMENT OF THE STOCHASTIC MODEL EQUATIONS

The first stage in the stochastic analysis of a ventilation system involves the development of the deterministic model equations. Following this, input parameters are treated as random variables and the equations converted into stochastic differential equations [SDE's] with the dependent variables, the contaminant concentrations, considered as stochastic processes. For discussion purposes, the development of the ventilation system model equations will be carried out with specific reference to a local exhaust ventilation [LEV] system with the understanding that the model can later be generalized.

2.1 *Idealizations and assumptions*

In order to produce a linear ordinary differential equation with time as the independent variable in the deterministic LEV system model, the following idealizations and assumptions are used.

(i) The spatial domain is partitioned into two communicating cells forming an open, one-dimensional system (Fig. 1). With reference to a LEV system, the inner cell, Cell 1, is representative of conditions occurring within the exhaust hood interior, and the outer cell, Cell 2, of the immediate workplace environment within which the exhaust hood is situated. Exterior to the outer cell, conditions are representative of the general environment with which the workplace interacts. The physical geometry is specified by the cell volumes ($\forall_1$, $\forall_2$) and the areas of surfaces (A_1, A_2) across which flows of contaminant and air occur.

(ii) Two species, air and a respirable contaminant, constitute a non-reacting mixture within the system. It will be assumed that the contaminant generation rate, $\dot{m}$,

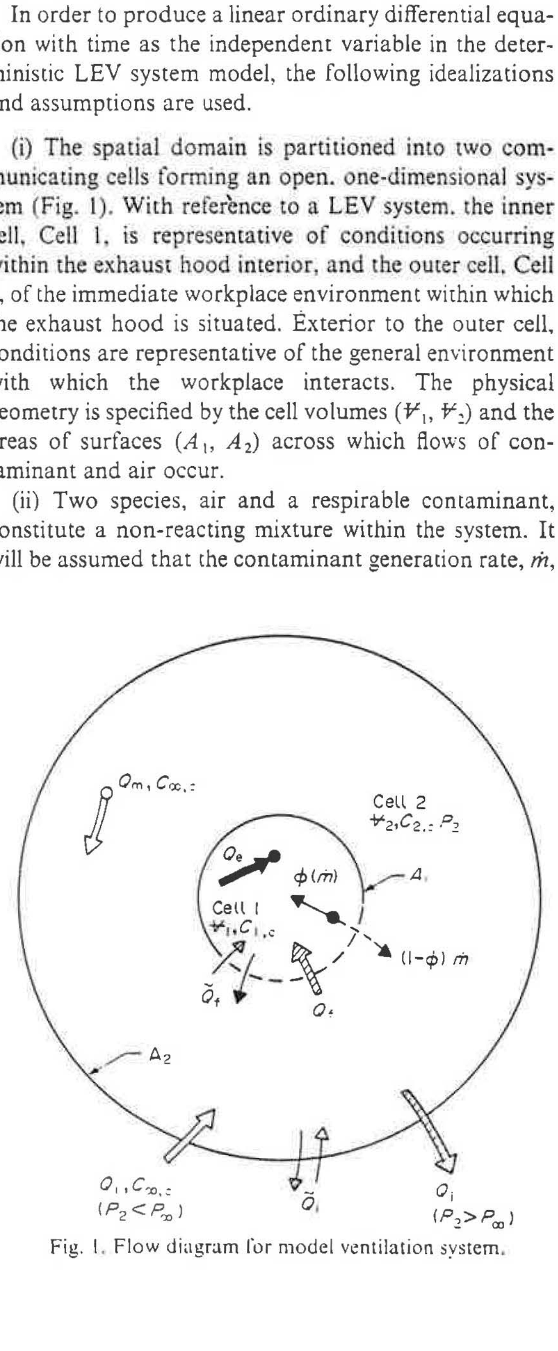

Fig. 1. Flow diagram for model ventilation system.

is sufficiently small that its contribution to the total mass in the system may be ignored. Thus, at any time the total mass in the system, or a portion of the system, is constant and equal to the contained air mass. This is equivalent to assuming that the mixture density, ρ, is equal to the density of air at the prevailing conditions. The contaminant is considered passive, meaning it is generated with negligible momentum and buoyancy, and is in the gas phase and/or is capable of being affected by air currents without momentum losses to the air flows.

(iii) Ventilation processes considered include both mechanical ventilation and infiltration–exfiltration processes. The influence of turbulence, unavoidable in most practical ventilation applications, will be included. Natural ventilation, such as that arising through open windows and stack effects is not considered explicitly, however, the model developed could be extended to include this case.

Infiltration–exfiltration processes are often characterized by leakage distributions and pressure differentials across an enclosing envelope. Results from [8] found that for all but particularly air-tight enclosures, a uniform leakage distribution gave the best estimate of flow across the envelope. Thus, within this work both mechanical ventilation and infiltration–exfiltration processes are approximated as one-dimensional.

(iv) To provide a model with behaviour similar to a real system, contaminant generation is treated as occurring within or at the outer boundary of Cell 1: that is, near the hood face. At the time of generation, contaminant may issue directly (undiluted by workplace air) into either Cell 1, Cell 2, or both. In the general case, ϕ is defined as the fraction of the contaminant release rate confined to Cell 1, and the compliment, $(1-\phi)$ as the fraction escaping undiluted into Cell 2. For the typical situation $\phi = 1$ and turbulence is the only mechanism for direct transport of contaminant from Cell 1 to Cell 2.

(v) The ventilation processes result in the transport of both air and contaminant into and out of the cells as well as between the cells. In principle, all the ventilation processes can be Reynolds decomposed into one-dimensional mean flows plus superposed turbulent processes, assuming the turbulence to be statistically stationary with respect to time.

With reference to Fig. 1, the one-dimensional mean flows are: Q_e, the exhaust hood *exhaust* or *extracted* flow, Q_f, the flow originating from the workplace that is drawn across the exhaust hood *face* and into the exhaust hood, Q_m, *makeup* air supplied as general dilution ventilation distributed throughout the workplace environment, and Q_i, *infiltration–exfiltration* flows occurring between the workplace and its exterior.

A study of the impact of infiltration transients, building structure dynamic response, and internal building air mass dynamic response on concentration levels in a fully-mixed one-zone model concluded that these parameters are unlikely to be of importance in a model of an infiltration—mechanical ventilation—air contamination system [9]. Therefore, to simplify the mathematical solution of the stochastic equations, all mean flows will be considered steady. It should be noted however, that this assumption is not a limitation dictated by the methods available for solving SDE's.

The effect of turbulence on contaminant transport between cells is included through explicit consideration of turbulent processes at *cell interfaces*. Superposed on the mean flows Q_f and Q_i are rapidly fluctuating (turbulent) flows characterized by the fluctuating velocity components v_f' and v_i' acting across the respective control surfaces. These are converted to turbulent flows $\tilde{Q}_f$ and $\tilde{Q}_i$ using a Boussinesq-like approach [10]. Molecular diffusion of contaminant across cell boundaries is assumed to be orders of magnitude less than that occurring as a result of turbulent transport and may be neglected.

The turbulent nature of the remaining flows in Fig. 1, are considered as contributing only to the mixing of contaminent and air within a cell and do not contribute to the transport of contaminant between cells, hence they do not appear explicitly in the transport equations developed in the next section.

(vi) Continuous spatial gradients of the contaminant concentration within a cell cannot be supported. This corresponds to an assumption of complete mixing and implies that the chemical composition of air within a cell is uniform. Heuristically it is argued that sufficient turbulent induced mixing occurs so that almost instantaneously, the contaminant concentration is constant throughout a cell. Finite changes in the contaminant concentration occur across cell boundaries. This is consistent with classical mixing theory models [11].

2.2 *Deterministic model equations*

The deterministic model equations for the temporal evolution of the contaminant concentration in Cell 1 and Cell 2 are now developed by: applying conservation of contaminant species to Cells 1 and 2; applying conservation of air species to Cells 1 and 2; combining the four mass balances, and finally non-dimensionalizing. Details of this procedure may be found in [12]. The final non-dimensional form of the deterministic ventilation model equations become:

$$\frac{dC_1^*(t^*)}{dt^*} + \kappa\left(\frac{Q_e}{Q_{e_{ref}}} + \frac{\tilde{Q}_f}{Q_{e_{ref}}}\right)(C_1^*(t^*) - C_2^*(t^*)) - \frac{\kappa\phi\dot{m}}{\dot{m}_{ref}} = 0, \quad (1)$$

$$\frac{dC_2^*(t^*)}{dt^*} - \frac{\tilde{Q}_f}{Q_{e_{ref}}}(C_1^*(t^*) - C_2^*(t^*)) + \left(\frac{Q_m}{Q_{e_{ref}}} + \frac{\tilde{Q}_i}{Q_{e_{ref}}}\right) \times (C_2^*(t^*) - C_\infty^*(t^*)) - (1-\phi)\frac{\dot{m}}{\dot{m}_{ref}} = 0. \quad (2)$$

In both equations, the first term represents the time variation of contaminant concentration in the cell. The second term in equation (1) accounts for the exhaust sink from Cell 1 and contaminant transport between the cells. The second term in equation (2) is due only to contaminant transport between the cells. The third term in equation (2) results from the environment source and contaminant transport by infiltration between the environment and Cell 2. The last source term in each equation arises from the fraction of contaminant emitted directly into the cell from the contaminant source. $Q_{e_{ref}}$ and $\dot{m}_{ref}$ are ensemble average values and κ is the cell volume ratio $\mathcal{V}_2/\mathcal{V}_1$.

2.3 *Stochastic interpretation of the ventilation system model equations*

Equations (1) and (2) form a coupled set of deterministic first order ordinary differential equations with independent variable t^*, and dependent variables $C_1^*(t^*)$ and $C_2^*(t^*)$. Seven *input parameters*, namely Q_e, Q_m, $\tilde{Q}_i$, $\tilde{Q}_f$, $C_\infty(t)$, $\dot{m}$, and ϕ, representing or associated with physical processes appear explicitly as either coefficients or source terms in these equations. These, together with the initial conditions, determine the specific behaviour of the solution. Considering that the environment within which the ventilation system operates is endowed with a large number of degrees of freedom that cause the physical processes in the environment to fluctuate rapidly, there is reason to treat the input parameters as stochastic in nature. Variability in these input parameters could be caused, for example, by the random activities of people situated within the ventilated environment, variations in the infiltration rate caused by changes in weather, variations in the contaminant generating processes, and variations in the hood face flow caused by reduction of the effective hood face area through the position of the worker.

In this study, the input parameters will be modelled as random variables of constant uncertainty. In a fashion analogous to the Reynolds' decomposition for turbulent quantities, the instantaneous random variables are decomposed as for example:

$$\dot{m}(t^*, \omega) = \bar{\dot{m}}(t^*) + \dot{m}'(t^*, \omega),$$

where the over-barred term is the "deterministic" mean (steady or unsteady) component of the physical process, the primed term represents the fluctuating or stochastic component, and ω is the probability variable. The stochastic components are modelled as independent stationary Gaussian white noise processes with zero mean and zero (auto and cross) correlations. For clarity in the following analysis the input parameters in equations 1 and 2 are grouped and renamed resulting in:

$$\frac{dC_1^*(t^*, \omega)}{dt^*} = K_1 C_1^*(t^*, \omega) + K_2 C_2^*(t^*, \omega) + K_3, \quad (3)$$

$$\frac{dC_2^*(t^*, \omega)}{dt^*} = K_4 C_1^*(t^*, \omega) + K_5 C_2^*(t^*, \omega) + K_6, \quad (4)$$

where the substitutions outlined in equations (5) have been used.

$$\left.\begin{aligned}
K_1 &= -\kappa\left(\frac{Q_e}{Q_{e_{ref}}} + \frac{\tilde{Q}_f}{Q_{e_{ref}}}\right)\\
K_2 &= -K_1\\
K_3 &= \frac{\kappa\phi\dot{m}}{\dot{m}_{ref}}\\
K_4 &= \frac{\tilde{Q}_f}{Q_{e_{ref}}}\\
K_5 &= -\left(\frac{\tilde{Q}_f}{Q_{e_{ref}}} + \frac{Q_m}{Q_{e_{ref}}} + \frac{\tilde{Q}_i}{Q_{e_{ref}}}\right)\\
K_6 &= \left(\frac{Q_m}{Q_{e_{ref}}} + \frac{\tilde{Q}_i}{Q_{e_{ref}}}\right)C_\infty^*(t^*) + (1-\phi)\frac{\dot{m}}{\dot{m}_{ref}}
\end{aligned}\right\} \quad (5)$$

Equations (5) are similarly decomposed into mean and stochastic components as:

$$K_i(t^*, \omega) = \bar{K}_i(t^*) + K_i'(t^*, \omega).$$

Substitution into the model equations (3) and (4) results in:

$$\frac{\mathrm{d}C_1^*(t^*)}{\mathrm{d}t^*} = \bar{K}_1 C_1^*(t^*) + \bar{K}_2 C_2^*(t^*) + \bar{K}_3 + K_1' C_1^*(t^*) + K_2' C_2^*(t^*) + K_3', \quad (6)$$

$$\frac{\mathrm{d}C_2^*(t^*)}{\mathrm{d}t^*} = \bar{K}_4 C_1^*(t^*) + \bar{K}_5 C_2^*(t^*) + \bar{K}_6 + K_4' C_1^*(t^*) + K_5' C_2^*(t^*) + K_6', \quad (7)$$

where the probability-based variable ω has not been written for clarity.

The final step required in the interpretation of the LEV system model equations as SDE's involves rewriting equations (6) and (7) in vector algebra notation and defining the mathematical representation of the stochastic components. The result is a vector stochastic differential equation describing the $C_1^*(t^*)$ and $C_2^*(t^*)$ processes which can be solved by the methods of stochastic calculus.

The set of stochastic differential equations (6) and (7) can be written as:

$$\frac{\mathrm{d}C_t^{*}*}{\mathrm{d}t^*} = \mathbf{F}(C_t^{*}*) + \mathbf{G}(C_t^{*}*)\xi_t*, \quad (8a)$$

or in differential form as:

$$\mathrm{d}C^* = \mathbf{F}(C_t^{*}*)\,\mathrm{d}t^* + \mathbf{G}(C_t^{*}*)\,\mathrm{d}W_t*, \quad (8b)$$

where:

$$\xi_t^T* = [K_1'K_2'K_3'K_4'K_5'K_6'], \quad (9)$$

$$C_t^{*}* = \begin{bmatrix} C_1^*(t^*) \\ C_2^*(t^*) \end{bmatrix}, \quad (10)$$

$$\mathbf{F}(C_t^{*}*) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \bar{K}_1 C_1^*(t^*) + \bar{K}_2 C_2^*(t^*) + \bar{K}_3 \\ \bar{K}_4 C_1^*(t^*) + \bar{K}_5 C_2^*(t^*) + \bar{K}_6 \end{bmatrix}, \quad (11)$$

$$\mathbf{G}(C_t^{*}*) = \begin{bmatrix} C_1^*(t^*) & C_2^*(t^*) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1^*(t^*) & C_2^*(t^*) & 1 \end{bmatrix}. \quad (12)$$

In keeping with the terminology of stochastic calculus, $C_t^{*}*$ is termed a vector (Itô or Stratonovich) stochastic process, $\mathbf{F}(C_t^{*}*)$ the drift matrix, and $\mathbf{G}(C_t^{*}*)$ the diffusion matrix. ξ_t* represents zero-mean independent Gaussian white noise processes and can be defined formally as the derivative in time of a Wiener process, W_t^*. Further detail, and specialized references on both Wiener and White noise processes may be obtained in [5] and [7].

Evaluation of equations (8) is conditional on a precise definition of the integral of the diffusion term, which is referred to as a stochastic integral. In this study, the Itô interpretation of the stochastic integral has been chosen to enforce conservation of mass in the mean equation.

3. THE STOCHASTIC SOLUTION PROCESS

Stochastic analysis allows the explicit calculation of an accurate but arbitrary solution trajectory for given sample functions of the initial values and of the Wiener processes. In general, however, there is less interest in arbitrary process sample functions or trajectory solutions than in certain properties of the solution process. Typically the solution process properties of interest would include a limited number of moments of the solution process, which for actual physical processes are often all that can be reasonably estimated. As evaluation of the solution process moments from the explicit analytical sample function solution is computationally difficult, the *Itô Lemma* is used here as a simpler means of obtaining the moment equations. A detailed presentation and proof of the *Itô Lemma* is contained in [5] and [6].

Application of the *Itô Lemma* to equation (8) will result in a system of solution process moment equations. For the general case of a *non-linear* ordinary SDE, the resulting system of moment equations involve an infinite hierarchy of differential equations with the equation for an nth order moment involving moments of higher order. Some form of closure, for instance truncation, must be adopted or else the defining SDE must be reformulated with linear approximations. For the special condition of a *linear* SDE, which equation (8) satisfies if certain constraints are imposed on K_3 and K_6, an nth order moment equation involves only moments of order n or less so that the moment equations can be integrated easily. In the case of a linear SDE the first and second order moments are sufficient to establish the mean solution and confidence intervals, while higher order moments are generally useful in non-linear formulations where they are used in closure schemes.

In the material that follows asterisks denoting non-dimensional time and concentration variables are omitted for clarity.

3.1 *Treatment of non-linear terms*

Equation (8) is in the strictest sense a non-linear SDE. The non-linearity arises in the two coefficient terms K_3 and K_6 of equations (5) through the variable products $\phi\dot{m}$, $C_\infty^* Q_m$, and $C_\infty^* \bar{Q}_i$. To avoid at least initially the necessity of closure schemes, the terms are linearized by assuming that ϕ and C_∞^* may be treated as constants. Physically, it is argued that in any well designed ventilation system ϕ, the fraction of contaminant release issuing directly into the hood, will be a constant equal to one and thus the only mechanism for contaminant transport into the room is through turbulence. In a similar fashion, C_∞^*, the ambient contaminant concentration, will be viewed as an infinite sink that is unaffectd by any contributions from the system source. These assumptions serve only as a first estimate. If in fact the physical system under consideration suggested that either or both ϕ and C_∞^* should be treated as random variables, or indeed stochastic processes, equation (8) would remain non-linear, and a closure scheme would be required.

3.2 *The first order moment equations*

Applying the *Itô Lemma* to equation (8), the two first order moment equations are obtained as:

$$\frac{d\bar{C}_1}{dt} = \bar{K}_1\bar{C}_1+\bar{K}_2\bar{C}_2+\bar{K}_3, \quad (13a)$$

$$\frac{d\bar{C}_2}{dt} = \bar{K}_4\bar{C}_1+\bar{K}_5\bar{C}_2+\bar{K}_6. \quad (13b)$$

The first order moment or mean equations of the solution processes are identical to that realized in a deterministic analysis. This was assured by the choice of the Itô interpretation of the model SDE. Solving for the transient case with time-independent coefficients ($\bar{K}_i(t) = \bar{K}_i$ = constant) by the method of undetermined coefficients:

$$\bar{C}_1(t) = B_1\left(\frac{m_1-\bar{K}_5}{\bar{K}_4}\right)e^{m_1 t}+B_2\left(\frac{m_2-\bar{K}_5}{\bar{K}_4}\right)e^{m_2 t}+\bar{C}_1(\infty), \quad (14a)$$

$$\bar{C}_2(t) = B_1 e^{m_1 t}+B_2 e^{m_2 t}+\bar{C}_2(\infty), \quad (14b)$$

where:

$$B_1 = \Delta\bar{C}_1\left(\frac{\bar{K}_4}{m_1-\bar{K}_5}\right)\left(1+\frac{m_2-\bar{K}_5}{m_1-m_2}\right)-\Delta\bar{C}_2\left(\frac{m_2-\bar{K}_5}{m_1-m_2}\right), \quad (15a)$$

$$B_2 = \frac{\Delta\bar{C}_2(m_1-\bar{K}_5)-\Delta\bar{C}_1\bar{K}_4}{m_1-m_2}, \quad (15b)$$

$$m_1 = \frac{\bar{K}_5+\bar{K}_1}{2}+\frac{1}{2}\sqrt{(\bar{K}_1-\bar{K}_5)^2+4\bar{K}_2\bar{K}_4}, \quad (15c)$$

$$m_2 = \frac{\bar{K}_5+\bar{K}_1}{2}-\frac{1}{2}\sqrt{(\bar{K}_1-\bar{K}_5)^2+4\bar{K}_2\bar{K}_4}, \quad (15d)$$

$$\Delta\bar{C}_1 = \bar{C}_1(0)-\bar{C}_1(\infty), \quad (15e)$$

$$\Delta\bar{C}_2 = \bar{C}_2(0)-\bar{C}_2(\infty). \quad (15f)$$

Equations (14a and b) are of the general form of an exponential buildup or decay in time of concentration in both Cell 1 and Cell 2.

3.3 *The second order moments*

The triplet of second order moment equations are obtained as:

$$\frac{d\bar{C}_1^2}{dt} = (2\bar{K}_1+\sigma_1^2)\bar{C}_1^2+\sigma_2^2\bar{C}_2^2+(2\bar{K}_2+2\gamma_{1,2}^2)\overline{C_1C_2}+2\bar{K}_3\bar{C}_1+\sigma_3^2, \quad (16a)$$

$$\frac{d\bar{C}_2^2}{dt} = \sigma_4^2\bar{C}_1^2+(2\bar{K}_5+\sigma_5^2)\bar{C}_2^2+(2\bar{K}_4+2\gamma_{4,5}^2)\overline{C_1C_2}+2(\bar{K}_6+\gamma_{5,6}^2)\bar{C}_2+\sigma_6^2, \quad (16b)$$

$$\frac{d\overline{C_1C_2}}{dt} = (\bar{K}_4+\gamma_{1,4}^2)\bar{C}_1^2+(\bar{K}_2-\gamma_{1,5}^2)\bar{C}_2^2+(\bar{K}_1+\bar{K}_5+\gamma_{1,5}^2-\gamma_{1,4}^2)\overline{C_1C_2}+\bar{K}_6\bar{C}_1+\bar{K}_3\bar{C}_2 \quad (16c)$$

where:

$$\sigma_1^2 = Var\{K_1\} = \left(\frac{Q_{e_{ref}}}{\mathcal{V}_2}\right)\left(\frac{\kappa}{Q_{e_{ref}}}\right)^2(Var\{Q_e\}+Var\{\tilde{Q}_f\}), \quad (17a)$$

$$\sigma_2^2 = Var\{K_2\} = \sigma_1^2, \quad (17b)$$

$$\sigma_3^2 = Var\{K_3\} = \left(\frac{Q_{e_{ref}}}{\mathcal{V}_2}\right)\left(\frac{\kappa\phi}{\dot{m}_{ref}}\right)^2 Var\{\dot{m}\}, \quad (17c)$$

$$\sigma_4^2 = Var\{K_4\} = \left(\frac{Q_{e_{ref}}}{\mathcal{V}_2}\right)\left(\frac{1}{Q_{e_{ref}}}\right)^2 Var\{\tilde{Q}_f\}, \quad (17d)$$

$$\sigma_5^2 = Var\{K_5\} = \left(\frac{Q_{e_{ref}}}{\mathcal{V}_2}\right)\left(\frac{1}{Q_{e_{ref}}}\right)^2 \times(Var\{\tilde{Q}_f\}+Var\{Q_m\}+Var\{\tilde{Q}_i\}), \quad (17e)$$

$$\sigma_6^2 = Var\{K_6\} = \left(\frac{Q_{e_{ref}}}{\mathcal{V}_2}\right)\left(\frac{C_\infty}{Q_{e_{ref}}}\right)^2(Var\{Q_m\}+Var\{\tilde{Q}_i\}), \quad (17f)$$

$$\gamma_{1,2}^2 = Cov\{K_1,K_2\} = -\left(\frac{Q_{e_{ref}}}{\mathcal{V}_2}\right)\left(\frac{\kappa}{Q_{e_{ref}}}\right)^2 \times(Var\{Q_e\}+\{Var\{\tilde{Q}_f\}), \quad (17g)$$

$$\gamma_{1,4}^2 = Cov\{K_1,K_4\} = -\left(\frac{Q_{e_{ref}}}{\mathcal{V}_2}\right)\left(\frac{\kappa}{Q_{e_{ref}}^2}\right)Var\{\tilde{Q}_f\}, \quad (17h)$$

$$\gamma_{1,5}^2 = Cov\{K_1,K_5\} = -\gamma_{1,4}^2, \quad (17i)$$

$$\gamma_{4,5}^2 = Cov\{K_4,K_5\} = -\left(\frac{Q_{e_{ref}}}{\mathcal{V}_2}\right)\left(\frac{1}{Q_{e_{ref}}}\right)^2 Var\{\tilde{Q}_f\}, \quad (17j)$$

$$\gamma_{5,6}^2 = Cov\{K_5,K_6\} = -\left(\frac{Q_{e_{ref}}}{\mathcal{V}_2}\right)\left(\frac{C_\infty}{Q_{e_{ref}}^2}\right) \times(Var\{Q_m\}+Var\{\tilde{Q}_i\}). \quad (17k)$$

Equations (16a – c) form a triplet of simultaneous first order linear differential equations in the three dependent variables $\bar{C}_1^2$, $\overline{C_1C_2}$, and $\bar{C}_2^2$.

Although in theory a general analytic solution can be obtained for the temporal evolution of the second order moments in the unsteady case, it is far more convenient and faster to resort to a numerical solution scheme unless specification of the coefficient terms results in a simplified equation set.

4. EXAMINATION OF SAMPLE CASES

The solutions obtained for the first and second order moment equations of the ventilation system model are applied in two sample cases. The effect of randomness or uncertainty of input-parameter values on the transient concentration statistics will be illustrated.

4.1 *Sample geometries*

Two ventilation systems will be modelled to demonstrate the flexibility of the two-zone model. The physical geometries, initial conditions, and input-parameter statistics (mean value and variance) are outlined in Tables 1 and 2.

The first system, referred to as Case 1, is a "typical" LEV system corresponding to a laboratory exhaust hood situation in an average-sized room. In the second system, Case 2, the properties of the system have been chosen to be characteristic of a "laminar" cross-flow clean room. In both sample cases, contaminant generation is confined to Cell 1 (ϕ = 1). For Case 1, this corresponds to a well-designed exhaust hood with a face area of A_1, and for

Table 1. Geometries and initial condition statistics for sample cases (note: σ = standard deviation)

Parameter	(Units)	Case 1	Case 2
$\mathcal{V}_1$	(m^3)	1.5	300
$\mathcal{V}_2$	(m^3)	300.0	300
A_1	(m^2)	1.5	30
A_2	(m^2)	220.0	220
$C_1(t=0)$	(ppm)	0	0
$C_2(t=0)$	(ppm)	0	0
$\sigma\{\bar{C}_1(t=0)\}$	(ppm)	0	0
$\sigma\{\bar{C}_2(t=0)\}$	(ppm)	0	0

which the only means of contaminant escape into Cell 2 is through turbulent processes. For Case 2, Cell 1 corresponds more generally to an "active" clean room region of cross-sectional area, A_1, and in which the contaminant generating processes are distributed. Cell 2, then, is to be identified as the zone of (worker) occupancy in Case 1, and in Case 2, as an inactive secondary region in which no contaminants are generated, located upstream of the active clean room region.

4.2 *Initial conditions*

For both sample cases the initial conditions in Cell 1 and Cell 2 have been fixed at zero with zero uncertainty. In a more general analysis, random initial conditions would be characterized by specifying mean initial concentration values and associated variances. Within the model, this corresponds mathematically to specifying initial values for both the first and second order moments.

Table 2. Parameter statistics for sample cases (note: σ = standard deviation)

Parameter	(Units)	Case 1	Case 2
C_∞	(ppm)	0	0
$\dot{m}$	($ml\ s^{-1}$)	50	50
$\frac{\sigma\{\dot{m}\}}{\dot{m}}$		0.25, 0.5	0.25, 0.5
ϕ		1	1
Q_f/A_1	($m\ s^{-1}$)	0.5	0.5
Q_f	($m^3\ s^{-1}$)	0.75	15
$\frac{\sigma\{Q_f\}}{Q_f}$		0.5	0.5
Q_e	($m^3\ s^{-1}$)	$Q_e = Q_f$	
Q_m	($m^3\ s^{-1}$)	$Q_m = 1.1Q_f$	
$\frac{\sigma\{Q_m\}}{Q_m}$		0.5	0.5
Q_i	($m^3\ s^{-1}$)	$Q_i = Q_m - Q_f$	
RMS(v_f')	($m\ s^{-1}$)	RMS(v_f') = $0.04+0.21(Q_f/A_1)$†	
RMS(v_i')	($m\ s^{-1}$)	RMS(v_i') = $0.04+0.21(Q_i/A_2)$†	
$\tilde{Q}_f$	($m^3\ s^{-1}$)	$\tilde{Q}_f$ = RMS(v_f')$A_1/2$†	
$\frac{\sigma\{\tilde{Q}_f\}}{\tilde{Q}_f}$		0.5	0.5
$\tilde{Q}_i$	($m^3\ s^{-1}$)	$\tilde{Q}_i$ = RMS(v_i')$A_2/2$†	
$\frac{\sigma\{\tilde{Q}_i\}}{\tilde{Q}_i}$		0.5	0.5

† [10]

4.3 *Contaminant source*

For illustrative purposes, gaseous ammonia emitted at a constant rate of 2.26 μkg mol s^{-1}, or roughly 50 ml s^{-1}, has been used for both sample cases.

4.4 *Infiltration–exfiltration processes*

For both sample cases, the Cell 2 exfiltration flows, $\tilde{Q}_i$, were "forced" by setting the make-up or supply air parameters, Q_m, at 110% of the exhaust flows, Q_e. This corresponds to the situation of a positively pressurized workplace. More sophisticated models of the infiltration–exfiltration processes can readily be accommodated through specification of wind loadings and building porosity and pressurization characteristics.

4.5 *Characterization of the RMS turbulence levels*

Calculations of $\tilde{Q}_i$ and $\tilde{Q}_f$ for both illustrative cases were based on a correlation of RMS turbulence levels with mean air velocities from [10], for mean air velocities equal to or greater than 0.05 m s^{-1}. Below this range, the RMS turbulence level was set equal to the mean value.

4.6 *Transient analysis*

The non-dimensional transient first order moment equations were solved exactly using equations (14a and b). Solutions for the non-dimensional transient second order moment equations (16a–c) were obtained using a fifth and sixth order Runge Kutta–Verner method. The first order moment equations were used directly in establishing mean concentrations, and the first and second order moment equations were used in determining the 95% confidence intervals ($\bar{C}_i \pm 2\sigma\{C_i\}$) of the solution processes. The results of the transient analysis for the two sample cases are illustrated in Figs 2–11, where each even-numbered figure contains model results assuming uncertainty or randomness in one input-parameter for Case 1 (LEV system) and the succeeding odd-numbered figure illustrates the analogous situation in Case 2 (clean room).

In each of the figures, the mean concentrations (solid lines) and associated 95% confidence intervals (broken lines) are illustrated in the upper diagram for Cell 1 and in the lower diagram for Cell 2. Curves with solid symbols are plotted against the left vertical axes scaled in dimensional (ppm) units, while curves with open symbols are plotted against the normalized non-dimensional scale of the right axes. Solutions are plotted against both dimensional time in seconds on the lower horizontal axes, and t^*, the non-dimensional time variable on the upper horizontal axes. In each figure, uncertainty or randomness has been assumed in only a single parameter with all other remaining parameters assumed to be constants of zero uncertainty. In this fashion, the sensitivity of the model to fluctuations or uncertainty in a particular parameter may be examined.

The mean concentration-time evolution characteristics are identical in Figs 2, 4, 6, 8 and 10 for Case 1 and Figs 3, 5, 7, 9 and 11 for Case 2, as they are unaffected in the Itô-type stochastic analysis by the stochastic nature of the input-parameters. Examination and comparison of

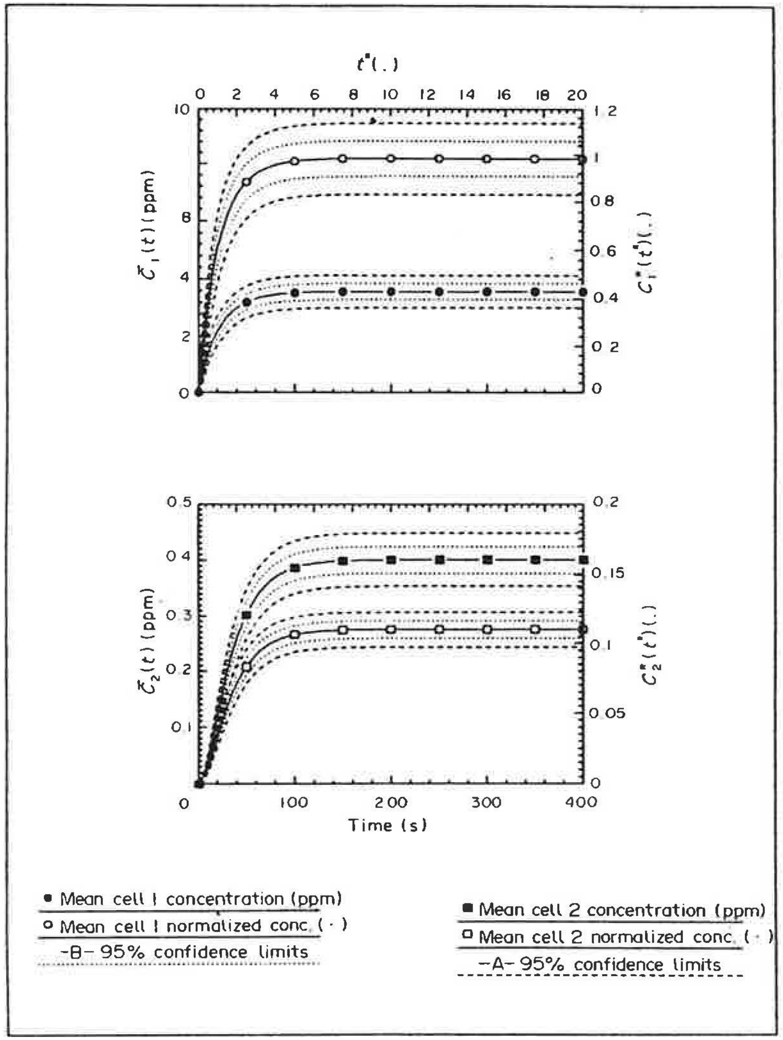

Fig. 2. Case 1: temporal evolution of cell concentration statistics with contaminant source strength uncertainty. Situation A $\sigma\{\dot{m}\} = 0.5\dot{m}$; situation B $\sigma\{\dot{m}\} = 0.25\dot{m}$.

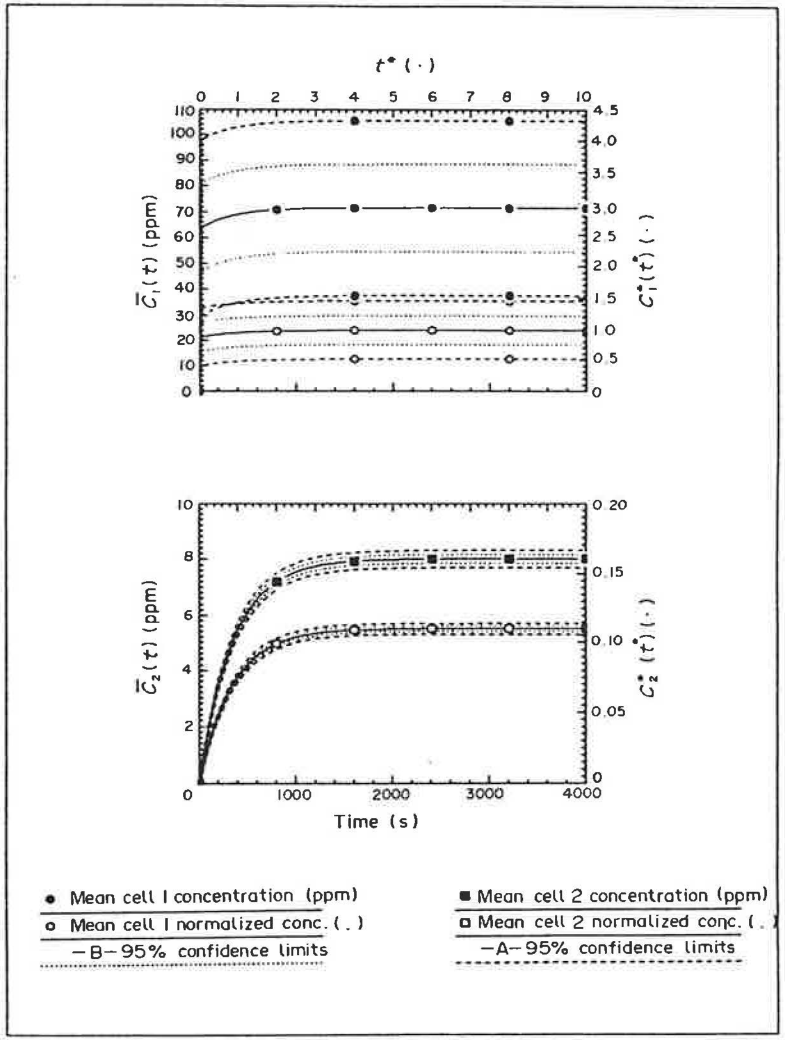

Fig. 3. Case 2: temporal evolution of cell concentration statistics with contaminant source strength uncertainty. Situation A $\sigma\{\dot{m}\} = 0.5\dot{m}$; situation B $\sigma\{\dot{m}\} = 0.25\dot{m}$.

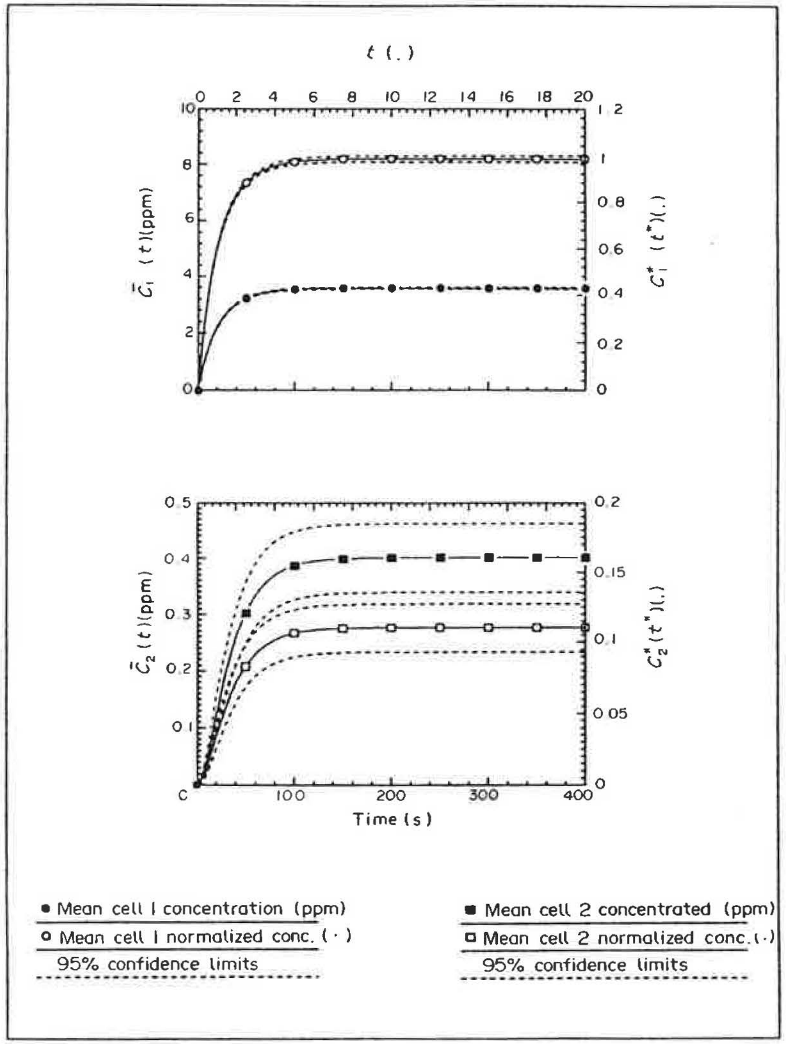

Fig. 4. Case 1: temporal evolution of cell concentration statistics with uncertainty in the level of inter-cell turbulence. $\sigma\{\tilde{Q}_t\} = 0.5\bar{\tilde{Q}}_t$.

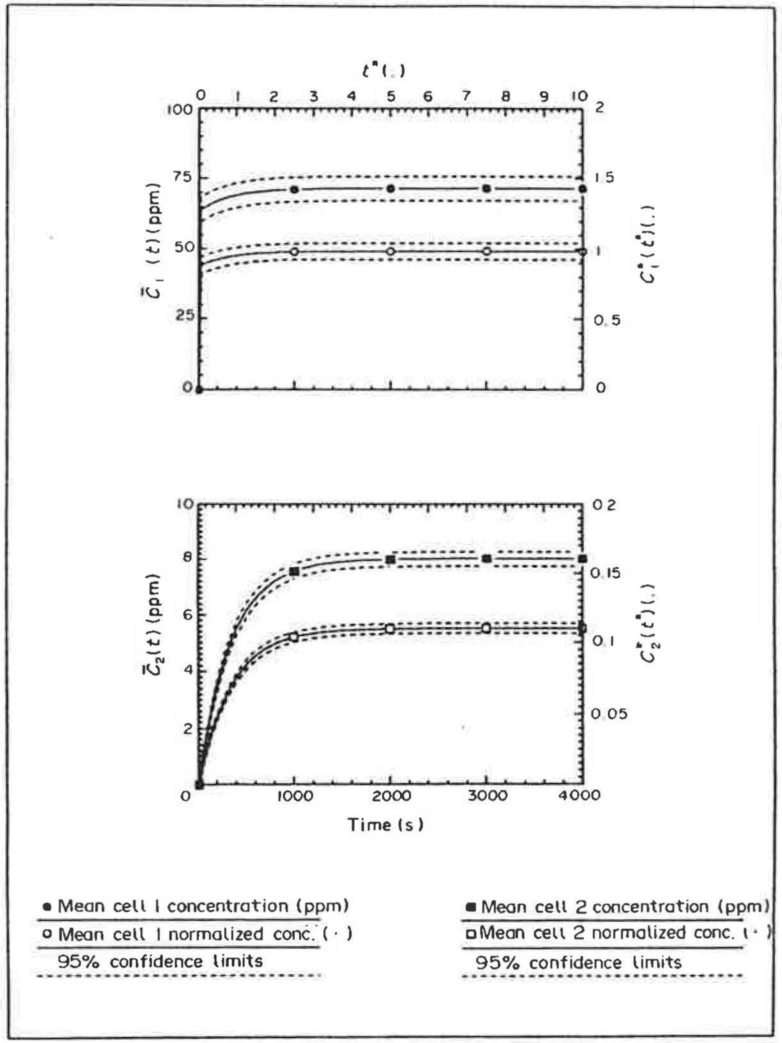

Fig. 5. Case 2: temporal evolution of cell concentration statistics with uncertainty in the level of inter-cell turbulence. $\sigma\{\tilde{Q}_t\} = 0.5\bar{\tilde{Q}}_t$.

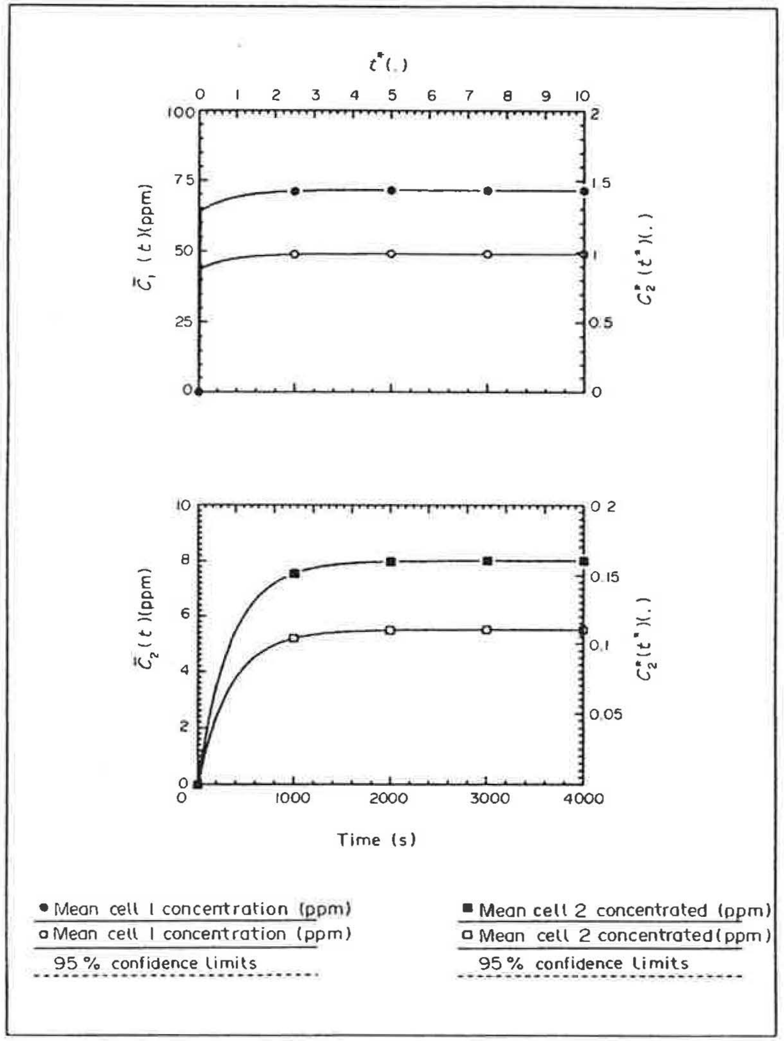

Fig. 6. Case 1: temporal evolution of cell concentration statistics with uncertainty in the level of turbulent exfiltration. $\sigma\{\tilde{Q}_i\} = 0.5\tilde{Q}_i$.

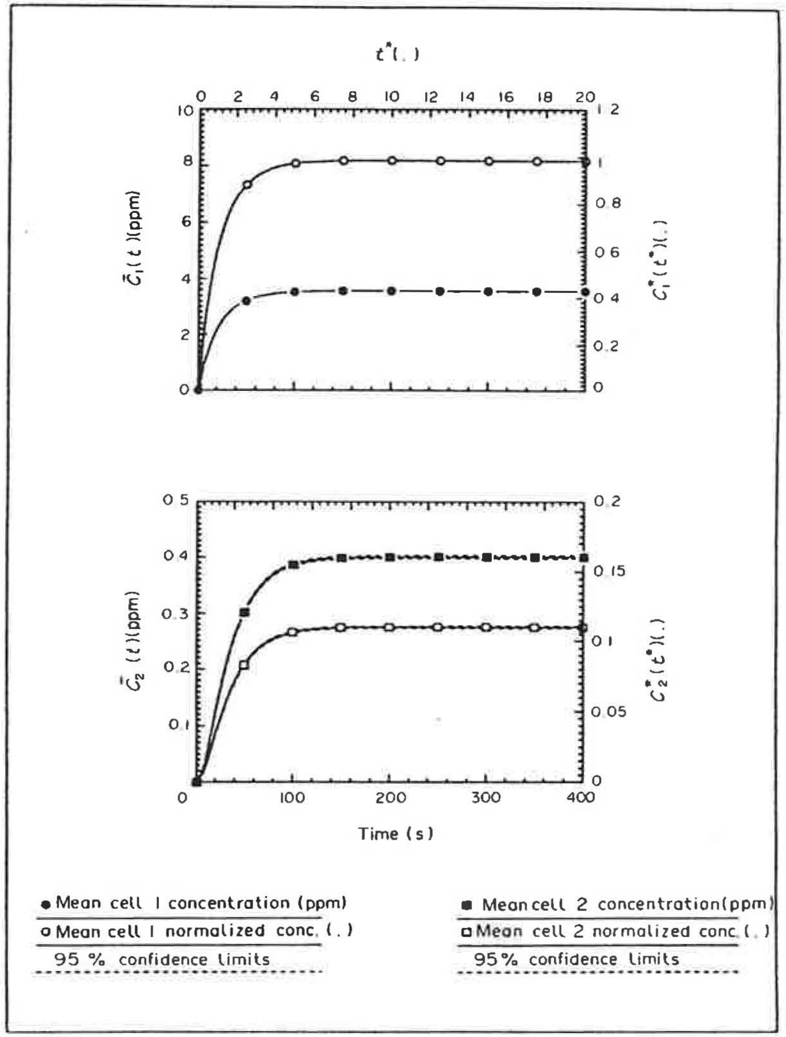

Fig. 7. Case 2: temporal evolution of cell concentration statistics with uncertainty in the level of turbulent exfiltration. $\sigma\{\tilde{Q}_i\} = 0.5\tilde{Q}_i$.

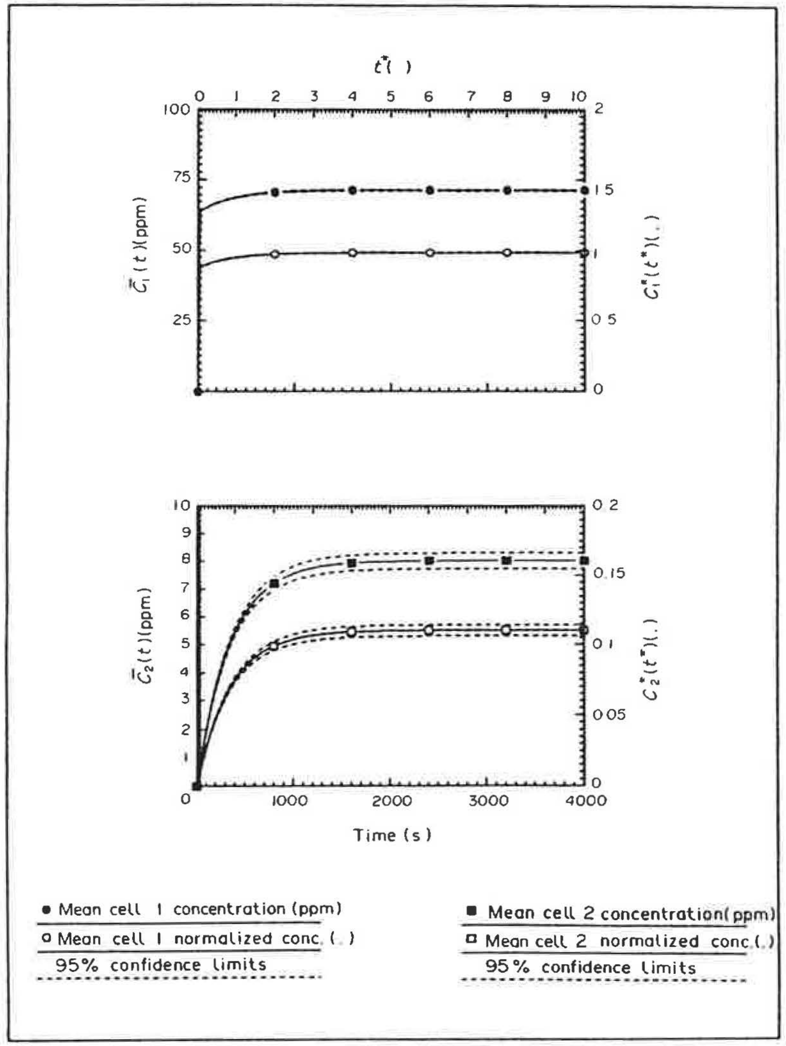

Fig. 8. Case 1: temporal evolution of cell concentration statistics with uncertainty in the level of make-up air. $\sigma\{Q_m\} = 0.5Q_m$.

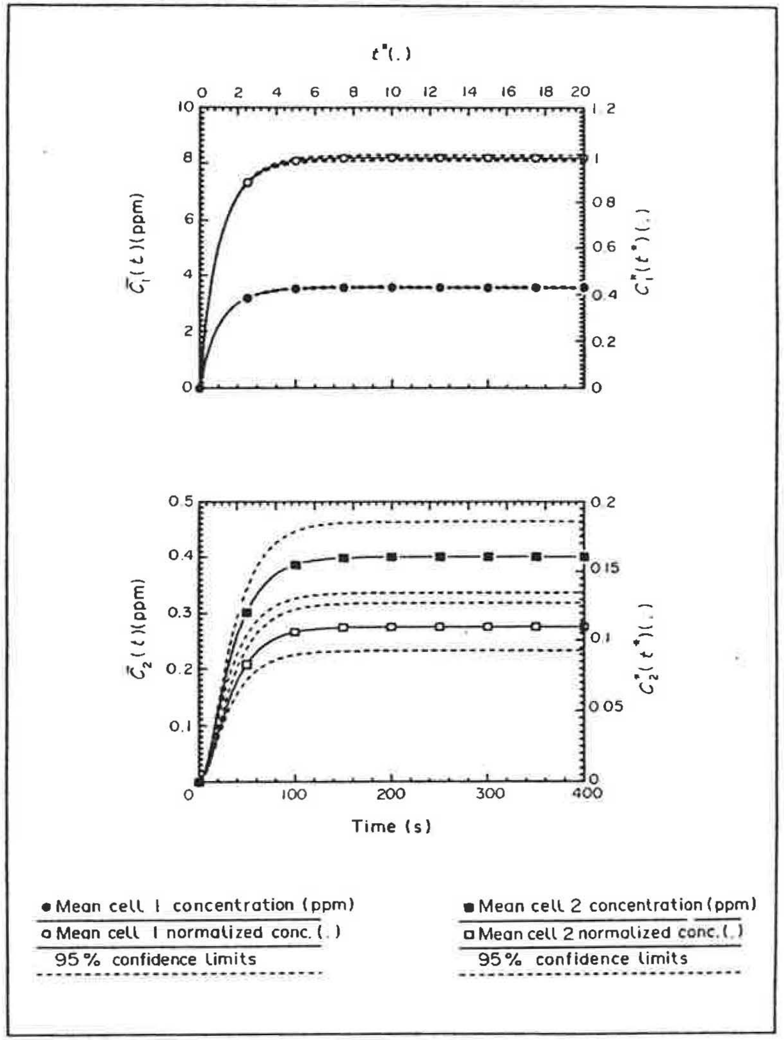

Fig. 9. Case 2: temporal evolution of cell concentration statistics with uncertainty in the level of make-up air. $\sigma\{Q_m\} = 0.5Q_m$.

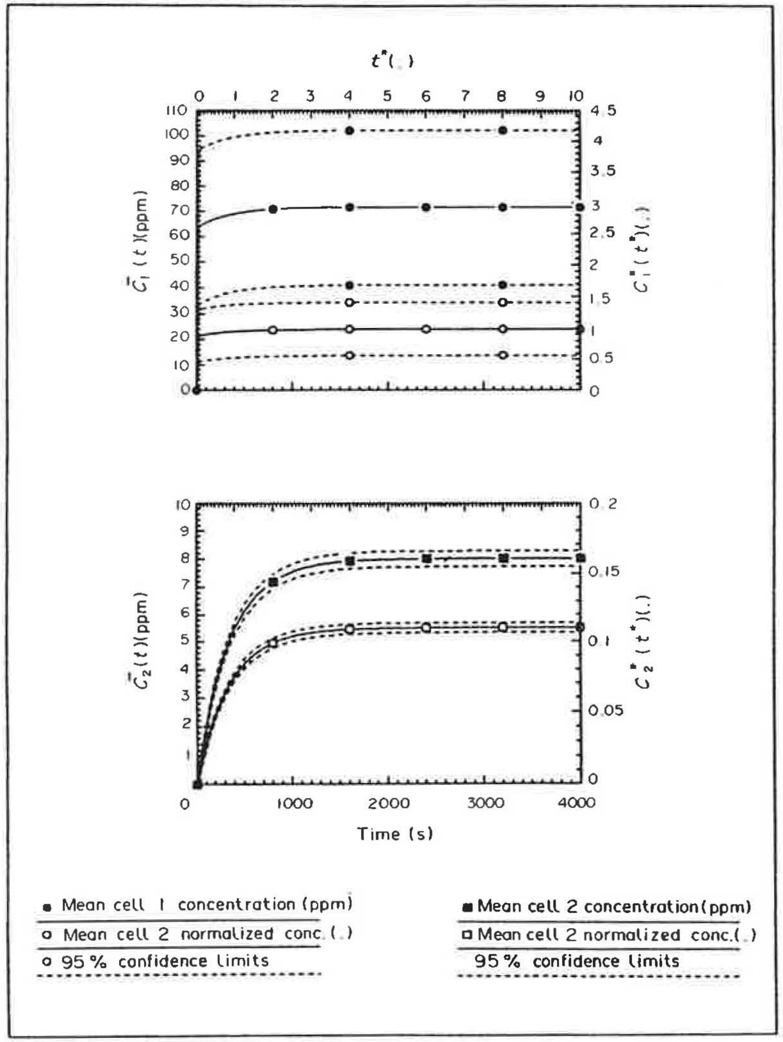

Fig. 10. Case 1: temporal evolution of cell concentration statistics with uncertainty in the level of exhaust air. $\sigma\{Q_e\} = 0.5Q_e$.

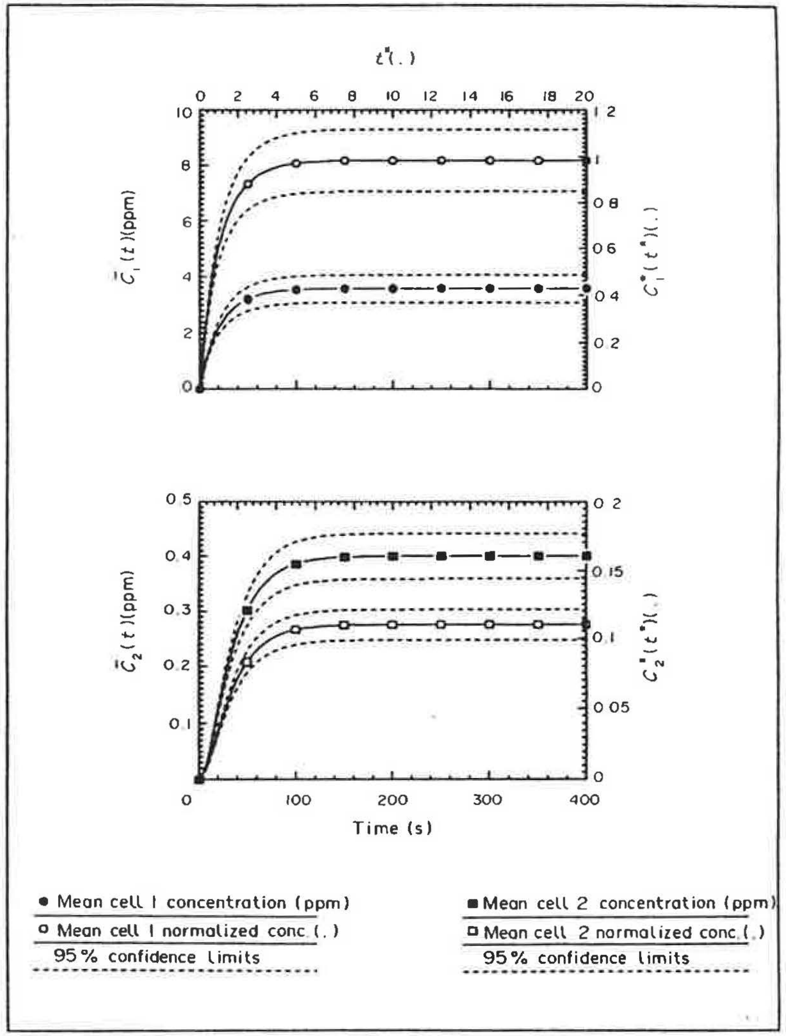

Fig. 11. Case 2: temporal evolution of cell concentration statistics with uncertainty in the level of exhaust air. $\sigma\{Q_e\} = 0.5Q_e$.

the two sample cases' mean concentration-time curves indicate that while for both cases, and for both cells, exponential-type growth of contaminant concentrations are predicted, the growth rates, as indicated by time to reach a fixed percentage of the steady-state value, are different. In both cases the supply zone, or Cell 1, approaches the steady-state contaminant level faster than Cell 2. In Case 1, 96% of the steady-state mean concentration is attained within 400 s for Cell 1, and 1200 s for Cell 2. For Case 2, the comparable times are 80 and 100 s respectively. Respective to Case 1, then, Case 2 demonstrates a faster system response to the contaminant source by reason of its respectively higher level of inter-cell contaminant transport processes.

The 95% confidence bounds on the solution processes arise from, and are a function of, the input-parameter uncertainty. They represent statistical limits on knowledge of the system state, or the contaminant concentrations, based on the uncertainty of the system inputs.

In Figs 2 and 3 where $\sigma\{\dot{m}\} = 0.5\dot{m}$ in situation A and $\sigma\{\dot{m}\} = 0.25\dot{m}$ in situation B, the large confidence interval bandwidths are evidence of the highly sensitive relationship of the cell concentrations with the instantaneous value of the source strength. For both cases, the model predicts that the supply zone, or Cell 1, concentrations are highly scattered about the mean values. The confidence intervals for Cell 2 are, as expected, narrower than those of Cell 1, but are still indicative of the overall sensitivity of the model predictions to source uncertainty.

In Figs 4 and 5, the impact of uncertainty in the inter-cell turbulent transport processes on concentration statistics are illustrated, where $\sigma\{\tilde{Q}_f\} = 0.5\tilde{Q}_f$. As this parameter represents the sole means of contaminant transport between the supply zone and Cell 2, the confidence interval bandwidth for Cell 2 is relatively broad. Case 1 also demonstrates a broad confidence interval in Cell 1, reflecting the high sensitivity of the hood concentration to addition or removal of contaminant mass given its small volume.

The effect of uncertainty in the turbulent exfiltration processes on concentration statistics are illustrated in Figs 6 and 7, where $\sigma\{\tilde{Q}_i\} = 0.5\tilde{Q}_i$. In both cases, due to the relatively small scale of the exfiltration processes, any uncertainty associated with the instantaneous level of $\tilde{Q}_i$ has almost negligible impact on realized concentrations.

Figures 8 and 9 illustrate model predictions with uncertainty in the make-up air, Q_m with $\sigma\{Q_m\} = 0.5Q_m$. In both cases the Cell 1 statistics demonstrate only marginal sensitivity to supply air uncertainty, while those of Cell 2 are more significant. Comparing Figs 8 and 9 with the respective results for variations in $\tilde{Q}_f$, it is interesting to note that the same order of magnitude confidence interval bandwidths are recorded for Cell 2 in both cases.

The final set of transient solutions are illustrated in Figs 10 and 11 for uncertainty in Q_e, the exhaust airflow, where $\sigma\{Q_e\} = 0.5Q_e$. In both cases, and for both cells, the impact of uncertainty in Q_e is comparable to that of uncertainty in the source strength, Figs 2 and 3. This can be explained by considering that the role of Q_e is to remove contaminant directly from Cell 1 out of the system, preventing any possiblity of it escaping into Cell 2. This is analogous to a reduction in the effective source strength.

5. CONCLUSIONS

The work carried out within this study represents a departure from the strictly deterministic or strictly empirical modes of thought in ventilation research by recognizing that many of the factors influencing the performance of a ventilation system are inherently random or variable—or at the very least—quantifiable only with uncertainty, and can be treated as random variables.

The stochastic modelling of a ventilation system provides a tool whereby quantitative statistical conclusions concerning the system performance may be drawn assuming the input parameters and initial conditions behave as random variables. Although the proposed model can be further extended in a straightforward manner to a multi-zone model, within the confines of the two cell analysis there is still sufficient flexibility to characterize a diverse range of ventilation systems.

Model predictions of the concentration statistics for two sample ventilation systems based on uncertainty of input-parameters have been illustrated and demonstrate the ability of the stochastic ventilation system model to provide quantitative statistical answers to practical design problems.

REFERENCES

1. F. Haghighat, P. Fazio and T. E. Unny, A predictive stochastic model for indoor air quality. *Bldg Envir.* **23**, 195–201 (1988).
2. F. Haghighat, M. Chandrashekar and T. E. Unny, Thermal behaviour of buildings under random conditions. *Appl. Math. Modelling* **11**, 349–356 (1987).
3. T. E. Unny and Karresha, Stochastic nature of output from conceptual reservoir model cascades. *J. Hydrology* **68**, 161–180 (1984).
4. D. L. Siurna and G. M. Bragg, Stochastic modelling of room air diffusion. *Proceedings of the Ventilation '85 Conference* (Edited by H. Goodfellow), Elsevier, New York (1986).
5. L. Arnold, *Stochastic Differential Equations: Theory and Applications*. Wiley, New York (1974).
6. A. H. Jazwinski, *Stochastic Processes and Filtering Theory*. Academic Press, New York (1970).
7. T. C. Gard, *Introduction to Stochastic Differential Equations*. Marcel Dekker, New York (1988).
8. Hiroshi Yoshino, F. Hasegawa and Y. Utsumi, Verification of calculation models of air infiltration using three types of test houses. 5th AIC Conference, Nevada (1984).
9. G. D. McIntosh, G. M. Bragg and D. L. Siurna, Transient modelling of indoor air quality. Trans-

actions of the Air Pollution Control Association Specialty Conference on Indoor Air Quality in Cold Climates (1985).

10. J. Thorshauge, Air-velocity fluctuations in the occupied zone of ventilated space. *Am. Soc. Heat Refrig. Air-Condit. Engrs Trans.* **88**, 753–764 (1982).
11. S. Nagata, *Mixing: Principles and Applications*. Wiley, New York (1975).
12. D. L. Siurna, A stochastic model of ventilation. Masters Thesis, University of Waterloo (1985).