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A Modified Form of the k-ϵ Model for Predicting Wall Turbulence

The high Reynolds number form of the k-ϵ model is extended and tested by application to fully developed pipe flow. It is established that the model is valid throughout the fully turbulent, semilaminar and laminar regions of the flow. Unlike many previously proposed forms of the k-ϵ model, the present form does not have to be used in conjunction with empirical wall function formulas and does not include additional terms in the k and ϵ equations. Comparison between predicted and measured dissipation rate in the important wall region is also possible.

1 Introduction

The k-ϵ model of turbulence has recently emerged as a powerful tool for prediction of many complex flow problems including jets, wakes, wall flows, reacting flows and flows with buoyancy, centrifugal, and Coriolis forces. The basis of the model is that it solves the transport equations for the turbulence energy and the isotropic turbulence dissipation rate. The set of model equations recommended by Launder and Spalding [1] for high Reynolds number flows has been most widely employed. For wall flows, these equations are normally used in conjunction with empirical wall function formulas. The success of this method depends on the "universality" of the turbulent structure near the wall and when disagreements are found between measurements and predictions, it is difficult to judge whether the weakness of the method lies in the basic model equations or in the wall function formulas.

Jones and Launder [2, 3] extend the original k-ϵ model to the low Reynolds number form which allows calculations right to the wall. Later, other forms of the k-ϵ model were developed for the same purpose, but still include extra terms in the transport equations in order to improve predictions in the wall region, or for computational expediency.

This paper describes the development of a new form of the high Reynolds number k-ϵ model, predictions with which are then compared with measurements for fully developed pipe flows.

2 The Proposed Model

It is assumed that turbulence can be characterized by the turbulence kinetic energy and the isotropic component of the turbulence energy dissipation rate which are determined from the transport equations

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\frac{\nu_t}{\sigma_k} + \nu_l\right)\frac{\partial k}{\partial x_j}\right] + \nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\frac{\partial U_i}{\partial x_j}\right) - \epsilon \qquad (1)$$

$$\frac{D\epsilon}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\frac{\nu_t}{\sigma_\epsilon} + \nu_l\right)\frac{\partial \epsilon}{\partial x_j}\right] + C_1 f_1 \nu_t \frac{\epsilon}{k}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - C_2 f_2 \frac{\epsilon^2}{k} \qquad (2)$$

The time-averaged flow field can be determined through the eddy viscosity given by

$$\nu_t = C_\mu f_\mu k^2/\epsilon \qquad (3)$$

where $\sigma_k = 1.0$, $\sigma_\epsilon = 1.3$, $C_1 = 1.44$, $C_2 = 1.92$, $C_\mu = 0.09$ as recommended in [1].

These equations are a general form of those given by Launder and Spalding [1] in which the functions f_1, f_2, and f_μ are all assumed to be identically unity. It has been found that this assumption cannot be valid within a laminar sublayer and appropriate functions must be chosen to ensure satisfactory predictions consistent with physical arguments. The following sections describe the development of these functions.

2.1 Development of the Function f_μ. For fully developed pipe flows, Rodi [4] has shown for $C_\mu = 0.09$ that f_μ is approximately 2.0 at the symmetry line and varies almost linearly with radius. However, the measurements used for evaluation of ν_t and k^2/ϵ are both subject to large errors and hence the calculated values of f_μ at any radius may not be reliable. Numerous successful applications of the high Reynolds number form of the k-ϵ model with wall function formulas suggest that f_μ should approximately equal unity in fully turbulent regions remote from solid walls. This is also consistent with the usual understanding of turbulence that properties should be fairly uniform in regions where viscous effects are small compared to turbulent ones. In regions very near a wall where viscous effects become important, properties will change rapidly and f_μ will also differ considerably from unity. Formulas for f_μ which have been proposed to account for wall effects include,

(i) $$f_\mu = \exp\left[-A_m/(1 + R_t A_t)\right] \qquad (4)$$

This is proposed by Jones and Launder [2, 3] with A_m and

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A_r given the values 2.5 and 0.02, respectively. Hoffman [5] later suggested that A_m should be 1.75 instead of 2.5. In either case, f_μ and ν_t will become a unique function of R_t. In this formulation f_μ is affected only indirectly by the presence of a wall through R_t.

(ii) $$f_\mu = R_k/(A_m + R_k) \quad (5)$$

This is proposed by Chien [6] with $A_m = 210.0$. f_μ is now directly dependent on the normal distance from a wall.

(iii) $$f_\mu = 1 - \exp(-A_m R_t) \quad (6)$$

This is proposed by Hassid and Poreh [7] with $A_m = 0.0015$. As in (i), f_μ and ν_t depend only on R_t.

The above formulas have been shown to yield good results when used with the forms of the k-ϵ model employed by the respective proponents. Not all of the above formulas for f_μ can, however, be used with a physically correct boundary condition for ϵ and the unmodified transport equations (1) and (2) as seen from the following argument.

At a wall ϵ is finite and $k = \partial k/\partial y = 0$. The variation of k and ϵ near the wall may be expanded in a Taylor series to give

$$k = p_2 y^2 + p_3 y^3 + \ldots$$
$$\epsilon = q_0 + q_1 y + q_2 y^2 + q_3 y^3 + \ldots$$

where the p's and q's are functions of the streamwise distances. At very small distances from the wall ν_t will be proportional to y^4, y^6 and y^8 if equations (4), (5), and (6), respectively, are used. Therefore, only equation (4) can provide near wall variation of ν_t consistent with the findings of Elrod [8] and the famous Van Driest formula

$$\nu_t \propto \nu_l y^{+2} (1 - e^{-y^+/26})^2 \frac{\partial U^+}{\partial y^+}$$

The assumption that ν_t is dependent only on R_t remains doubtful as it seems plausible that the presence of a wall should have a direct influence on ν_t such as that given in equation (5). Hence, an alternative formula is required and can be obtained by using the succcessful Hassid-Poreh one-equation model employed by Gibson, et al. [9] in which ν_t and ϵ are given by

$$\nu_t = 0.2274 y k^{1/2} (1 - e^{-0.01189 R_k}) \quad (7)$$

$$\epsilon = 0.4 \frac{k^{3/2}}{y} (1 - e^{-0.01189 R_k}) + 2.0 \nu_l \frac{k}{y^2} \quad (8)$$

Treating R_k as a separate independent variable, equations (7) and (8) can be combined to eliminate y thus expressing ν_t in terms of k, ϵ, R_k and ν_l:

$$\nu_t = \frac{0.09}{2} \frac{k^2}{\epsilon} (1 - e^{0.01189 R_k})^2 \left[1 + \sqrt{1 + 50.0 \frac{\nu_l \epsilon}{k^2} (1 - e^{-0.01189 R_k})^{-2}} \right] \quad (9)$$

Comparison with equation (3) for $C_\mu = 0.09$ shows that

$$f_\mu = 0.5 (1 - e^{0.01189 R_k})^2 \left[1 + \sqrt{1 + 50.0/[R_t (1 - e^{-0.01189 R_k})^2]} \right] \quad (10)$$

Instead of equation (10), the simpler relationship of equation (11) is postulated

$$f_\mu = (1 - e^{-A_\mu R_k})^2 \left(1 + \frac{A_t}{R_t} \right) \quad (11)$$

which still renders f_μ a function of both R_k and R_t and ν_t is proportional to y^4 near the wall. The presence of a wall now has a direct and an indirect influence on f_μ. For high turbulence levels, f_μ will tend to unity at large distances from the wall but retain the R_k dependence near the wall. The values of the constant A_μ and A_t are determined by trial and error.

2.2 Development of the Function f_1. Computations with the high Reynolds number form of the model with wall function formulas suggest that f_1 is approximately unity remote from a wall. In the near wall region it is found that f_1 assumes larger values in order to increase the predicted dissipation rate thereby reducing the predicted turbulence level to match available experimental data. If f_1 is held constant and equal to unity, additional destruction terms [2, 3, 5–7] would be required in the k-transport-equation to yield reasonable predictions. It is proposed that

$$f_1 = 1 + (A_{C1}/f_\mu)^3 \quad (12)$$

Nomenclature

A_m, A_r, A_μ, A_{C1}, A_t = turbulence model constants
C_1, C_2, C_μ = turbulence model constants
D = pipe diameter
f_1, f_2, f_μ = turbulence model functions
k = turbulence energy = $\frac{1}{2}\overline{u_i' u_i'}$
n = power law index
Re = Reynolds number $\frac{U_b D}{\nu_l}$
R_k = turbulence Reynolds number $\frac{k^{1/2} y}{\nu_l}$
R_t = turbulence Reynolds number $\frac{k^2}{\nu_l \epsilon}$
r = radial coordinate
U = time-averaged axial velocity
U_b = bulk velocity
U^* = shear velocity $\sqrt{\tau_w/\rho}$
U^+ = ratio of U/U^*
u' = instantaneous fluctuating component of axial velocity
U_i, U_j = tensor notation for velocities in the i and j directions, respectively
v' = instantaneous fluctuating components of radial velocity
x_i, x_j = tensor notation for space coordinates
y = normal distance from wall
y^+ = distance from wall defined as $U^* y/\nu_l$
ϵ = isotropic turbulence dissipation rate = $\nu_l \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}$
ϕ = generalized variable
ν = viscosity
σ_k = diffusion Prandtl number for turbulence energy
σ_ϵ = diffusion Prandtl number for dissipation rate
τ_w = shear stress at the wall

Subscripts

eff = effective
j = j[th] node
l = laminar
max = maximum
t = turbulent
w = at wall

Superscripts

$'$ = instantaneous fluctuating component
$-$ = time mean value

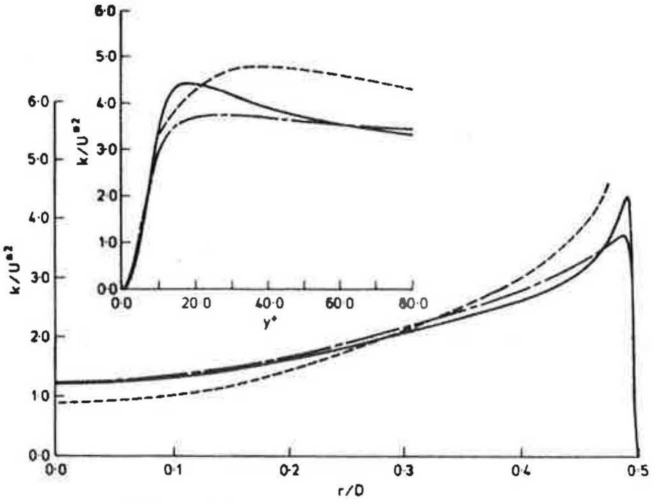

Fig. 1(a) turbulence kinetic energy

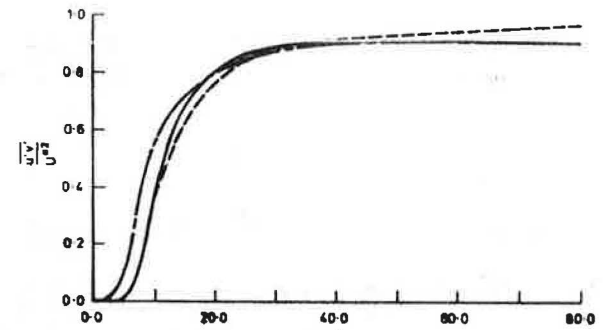

Fig. 1(b) turbulence shear stress
____________present model, Re = 41667
___ _ ___Jones and Launder's model, Re = 41667
\-\-\-\-\-\-\-\-\-\-\-\- measurement, Hinze [12], Re ≃ 41000

Fig. 1 Comparison for various turbulence quantities

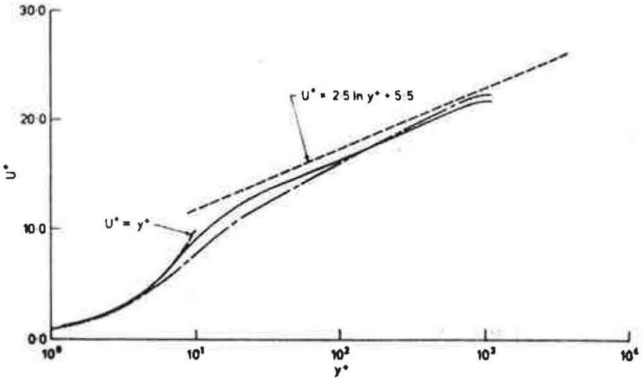

Fig. 2 Variation of mean velocity, Re = 41667
____________present model
___ _ ___prediction with Jones and Launder model

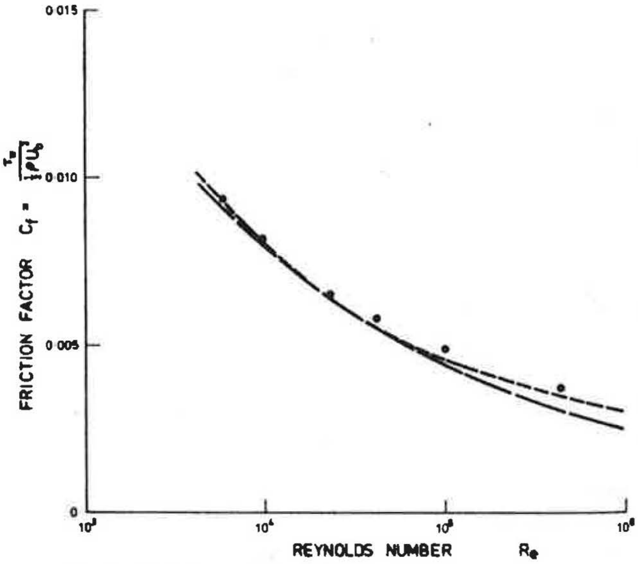

Fig. 3 Variation of friction factor with Reynolds number
⊙ present model, computed points
___ ___ ___ Blasius [11]
\-\-\-\-\-\-\-\-\-\-\-\- Nikuradse [11]

which makes f_1 a function of f_μ only. The constant A_{C1} should have a small value so that remote from a wall and when the turbulence level is high, f_μ and hence f_1 will be approximately unity. Close to a wall, f_μ will be small but finite and f_1 will become large. The value of the constant A_{C1} is again obtained by trial and error.

2.3 Development of the function f_2. Low Reynolds number forms of the k-ϵ model [2, 3, 6-8] employ the assumption

$$f_2 = 1 - 0.3e^{-R_t^2} \tag{13}$$

Since ϵ and its derivatives $\partial\epsilon/\partial x_j$ and $\partial^2\epsilon/\partial x_j^2$ are not infinite at a wall, f_2 must tend to zero as R_t tends to zero. Hence, equation (13) is not directly applicable but can be rectified by omitting the factor 0.3 in front of the exponential term, that is

$$f_2 = 1 - e^{-R_t^2} \tag{14}$$

2.4 Boundary Condition for ϵ. Since $\nu_t = \partial k/\partial r = 0$ at the wall and ν_l is constant, equation (1) reduces to $\epsilon_w = \nu_l(\partial^2 k/\partial r^2)_w$, the finite difference analog of which is obtained from a Taylor series expansion of k about the wall.

3 Application of the Model

3.1 Optimum Values of the New Constants A_μ, A_t and A_{C1}. Calculations were performed [10] for different Reynolds numbers by using a finite difference scheme with successive over-relaxation. For most cases, 91 grid points and a relaxation factor of 1.51 were used together with the convergence criterion

$$|(\phi_j^{(N)} - \phi_j^{(N-1)})/\phi_j^{(N)}| \leq 0.0001$$

where $\phi_j^{(N)}$ is the value of ϕ at the j^{th} grid node after the N^{th} iteration cycle and $\phi_j^{(N-1)}$ that for the $(N\text{-}1)^{\text{th}}$ iteration cycle. Best agreement with the measurements of Nikuradse (as reported in [11]) and those of Laufer reported in [12] was obtained for $A_\mu = 0.0165$, $A_t = 20.5$, and $A_{C1} = 0.05$.

3.2 Comparison of Predictions With the Present Model and Previous Data. Typical predictions for various turbulence quantities are compared in Figs. 1(*a*) and (*b*) with the model of Jones and Launder using the present finite difference scheme, and with Laufer's measurements made at Re = 41000 as reported by Hinze [12], although, the extrapolated experimental results for $\overline{u'v'}$ at small y^+ have not been shown in Fig. 1(*b*). This is because of the lack of reliable measurements to permit correct extrapolation to meet the known limiting derivatives of $\overline{u'v'}$ at the wall, namely, $\partial[\overline{u'v'}]/\partial y = 0$ and $\partial^2\overline{u'v'}/\partial y^2 = 0$. The present model meets both these values as well as Elrod's [8] limiting

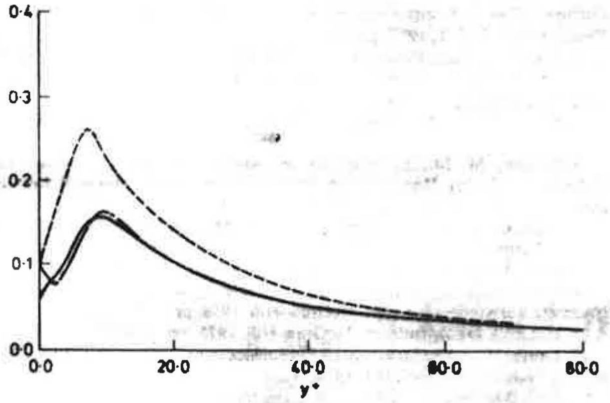

Fig. 4(a) Variation of dissipation rate near the wall
———— present model, Re = 41667
— - — prediction with equation (16) for f_1
------------ measurement, Hinze [12] Re = 41000

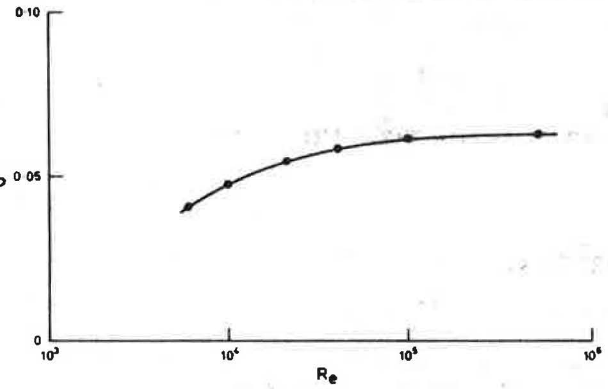

Fig. 4(b) Dissipation rate at the wall as a function of Reynolds number
-⊖- present model, computed points

relationship of $\overline{u'v'} \propto y^4$. Predicted velocity distributions using the two models are also compared with the universal velocity profile, Fig. 2. It can be seen that the present model is at least equal to, if not better than that of Jones and Launder.

Comparisons of the friction factor C_f from Nikuradse's data [11] and the Blasius formula

$$C_f = 0.079(\mathrm{Re})^{-0.25} \tag{15}$$

are shown in Fig. 3 for $6000 \leq \mathrm{Re} \leq 417000$. At low Reynolds numbers (Re < 30000), the agreement is excellent but for higher Reynolds numbers, predicted values appear to be slightly too high. To obtain C_f, full solutions of the equations were obtained [10] at all Re of Fig. 3. For the Re range covered, distributions are similar to those of Figs. 1 and 2.

3.3 Wall Dissipation Rate. The computed near wall variation of dissipation is compared in Fig. 4(*a*) with measurements as reported by Hinze [12]. A discrepancy of about 45 percent is noted at the wall but such a measurement is subject to larger errors. It is shown in [10] that the wall dissipation rate estimated by Hinze from Laufer's data [13] is approximately 1.9 times the correct value, thus favoring the present prediction.

It is also noted that the computed profile for ϵ shows a small "kink" at a $y+$ value of about 5. During an early stage in the development of the present model, it was found that a pronounced "kink" can be produced by choosing a different function for f_1. For example the broken line shown in Fig. 4 was obtained using

$$f_1 = 1 + \left(\frac{0.05}{f_\mu}\right)^2 \tag{16}$$

with $A_\mu = 0.04$, $A_t = 20.54$ and Re = 23300. This is in line with the recent measurements of Schildknecht, et al. [14] who have demonstrated that the measurable portion of ϵ does exhibit such a phenomenon. Until more information for dissipation rate is available, it will be difficult to assert positively the relationship between f_1 and f_μ but it appears that f_1 should be large when f_μ is small, and that f_1 may take the form of

$$f_1 = 1 + \left(\frac{A_{C1}}{f_\mu}\right)^n \tag{17}$$

thus expressing the hypothesis that enhanced dissipation rate near a solid boundary is associated with an increased damping of turbulence length scales.

The dissipation rate at the wall as a function of Reynolds number computed with the present model is shown in Fig. 4(*b*). No experimental data for comparison appear to be available.

3.4 Sensitivity Test. The aim of this test is to study the sensitivity of the solutions with respect to changes in the new constants A_μ, A_t, and A_{C1}. The test was carried out at Re = 23300 for $A_\mu \pm 5$ percent and A_{C1} and $A_t \pm 10$ percent about the optimum values of Section 3.1. Detailed results are available in [10] and give changes in C_f of 2.6 percent and −4.0 percent, −3.7 percent and 5.6 percent, and 0.4 percent and −2.0 percent, respectively. Although the predicted friction factor can change by a few percent by varying the constants, the predicted distributions of variables were found not to exhibit noticeable differences except for the dissipation rate, in which the "kink" becomes more pronounced with decreased A_{C1}.

4 Conclusions

A new form of the k-ϵ model has been developed and tested by application to fully developed turbulent pipe flow. The present model is defined by the set of equations (1)-(3). The values of the constants C_μ, C_1, C_2, σ_k, and σ_ϵ have been established by previous workers. The functions f_μ, f_1, and f_2 have the forms given by equations (11), (12), and (14), respectively. The values of the additional constants A_μ, A_t, and A_{C1} are reported in Section 3. Satisfactory predictions have been obtained with the present model and agreement with available experimental data is found to be good. It is also found that the predicted distributions of variables are not very sensitive to small changes in the values of the constants A_μ, A_t, and A_{C1}.

It is established that the present model is valid throughout the fully turbulent, semilaminar and laminar regions in fully developed pipe flows. In contrast to previous forms of the high Reynolds number k-ϵ model, the present one does not require the use of wall function formulas and does not require the introduction of additional terms into the transport equations. The present model also allows a study of predicted versus measured ϵ in the important wall region to be made. Computed profiles for dissipation rate show a small "kink" near the wall which agrees with recent measurements. It is demonstrated that the size of the kink can be altered by adjusting the values of n and A_{C1} in equation (17) but firmer recommendations must await the availability of reliable data for dissipation rate near the wall.

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