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An Idealised Model for Room Radiant Exchange

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In order to arrive at a simple design procedure to handle heat transfer in a room, certain fairly complicated expressions for radiant exchange are needed, and moreover the further operations that have to be performed on them are involved and lead to non-exact relations. The corresponding operations for radiant exchange between patches which form part of a spherical cavity are elementary and exact, and lead to results which are quite close to those for a cubic enclosure. Thus a consideration of radiant exchange in a sphere illustrates the principles of radiant exchange in a room, elegantly and with little of its complexity.

INTRODUCTION

IN AN EARLIER paper, the present author advanced a series of ideas relating to radiant exchange in a room.

(1) The 15 conductances expressing the direct exchange of radiation between the six black body surfaces of a rectangular room can be replaced by a set of 6 conductances, each linking a surface to a central "radiant star" node, notated as T_{rs}. The external effect of the star-based system cannot be exactly equivalent to that of the surface-surface set, but using an optimal procedure based on least squares methods, the equivalence can be made quite close and is virtually exact for a cube.

(2) The radiant star node T_{rs} is a fictional construct and longwave radiation strictly speaking cannot be input at it. If however the radiant output Q_r from an internal heat source is taken to be input at T_{rs}, the temperature it generates there is a fair approximation to the actual space averaged observable radiant temperature T_{rv} in the room. In fact if the source is placed centrally in the room, T_{rs} is some 14% lower than T_{rv}, but if the source is placed at the wall—a more realistic position—the two are more nearly equal. It will be assumed for design purposes that the fictitious value T_{rs} provides an adequate estimate of the physically meaningful volume-average radiant temperature T_{rv}.

(3) The effect of the emissivity of each surface can be included in the star network. The conductance linking T_{rs} to the temperature node T_j of surface J can then be conveniently written as $A_j \cdot E_j \cdot h_r$, where A_j is the area of the surface, E_j a dimensionless factor which expresses both the geometrical aspect of surface J in relation to the enclosure, and the effect of its emissivity; h_r is around 5.7 W m^{-2} K^{-1} at room temperatures.

(4) It is routinely assumed that the convective component Q_a of the heat input from an internal hot body source can be taken as input at the volume-averaged air temperature T_{av}, and that a convective conductance of type $A_j \cdot h_{cj}$ links T_{av} to T_j. The internal exchange of heat within the room can thus be expressed in terms of two independent star-based networks, one radiative, the other convective, with heat being input at each of the two nodes. The author has described this as the "binary star model". Comfort temperature T_c can be expressed explicitly in this formulation and is taken as a node linked to T_{rs} and T_{av} by conductances which are very small compared with typical values of the enclosure conductances $A_j \cdot E_j \cdot h_r$ and $A_j \cdot h_{cj}$.

(5) A one star model, the "rad–air" model can be derived from the binary star model. T_{av} retains its function. T_{rs} is no longer present, but is replaced by the rad–air node T_{ra}; T_{ra} is a linear function of T_{rs} and T_{av}. There is a very large conductance between T_{ra} and T_{av}. The radiant input Q_r from an internal source has now to be treated as an augmented input at T_{ra} together with a withdrawal of the excess at T_{av}. In special circumstances, the rad–air and binary star models are exactly equivalent to each other; in general this will not be so, but the equivalence is normally expected to be sufficiently close for design use.

Comfort temperature cannot be explicitly included in the rad–air model but it is easily calculated. If a single star model for room internal heat exchange is to be adopted, as it usually is for the purpose of design of heating and cooling equipment, the rad–air model is the logically appropriate approach. T_{ra} is the indoor equivalent of the sol–air temperature T_{sa} which serves to combine the effects of ambient air temperature and absorbed solar gain at wall exterior surfaces.

Stages 1–3 of this argument are concerned with radiant exchange alone and the first two involve some

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cumbersome view factor expressions and the solution of simultaneous equations in order to arrive at an optimal but of course, non-exact, equivalence of the surface–surface and surface–star point networks. It is the purpose of this paper to show that in the case of radiant exchange in a sphere, these stages can be conducted very simply and exactly, and that they lead to results very similar to those for a rectangular room. A treatment of radiant exchange in a sphere in fact provides a compact and elegant introduction to room radiant exchange.

2. RADIANT EXCHANGE IN A SPHERICAL CAVITY

We have to show how the radiant exchange between areas which have the form of patches on the surface of a spherical cavity can be expressed as an exchange via a central radiant star node, T_{rs}.

2.1. *Surface–surface exchange*

Consider two such very small patches, δA_1 and δA_2. The view factor F_{12} is the fraction of radiation leaving δA_1 diffusely that is intercepted by δA_2. It is routinely given by the expression:

$$F_{12} = \cos\theta_1 \cdot \cos\theta_2 \cdot \delta A_2/\pi r^2, \quad (1)$$

where r denotes the distance between the areas and θ is the angle between the normal to the area and the direction of r.

Suppose that the sphere is of radius R and that the two patches are situated an angular distance 2ϕ apart, seen from the centre of the sphere. In this case, $\theta_1 = \theta_2 = \pi/2 - \phi$, and $\frac{1}{2}r = R \cdot \sin\phi$. It follows that:

$$F_{12} = \delta A_2/4\pi R^2 \quad \text{or} \quad \delta A_2/S, \quad (2)$$

where S is the area of a sphere, $4\pi R^2$.

The geometrical conductance δC_{12} between the two patches is given as:

$$\delta C_{12} = \delta A_1 \cdot F_{12} = \delta A_2 \cdot F_{21} = \delta A_1 \cdot \delta A_2/S. \quad (3)$$

Clearly, the conductance from area 1 to a finite area 2 is found by summing adjacent small areas and conversely so the conductance between any two finite areas on a sphere is given by:

$$C_{12} = A_1 \cdot F_{12} = A_2 \cdot F_{21} = A_1 \cdot A_2/S. \quad (4)$$

The expression for the view factor is noted by Howell and Siegel [1]. It is very simple, comparable in simplicity with that for the surfaces of a long corridor as given by Hottel's crossed strings approach [2]. By contrast, the exact view factors for surfaces in a rectangular room are relatively complicated. The present author [3] however has shown that they lead to an approximate expression for values of F_{jk} in a room which has the same form as the value of F_{jk} for a sphere [i.e. equation (4)]: values for the surface-to-star point conductances, which are approximately proportional to area, replace the areas themselves.

The physical conductance (units, W K^{-1}) between areas A_1 and A_2 at temperatures T_1 and T_2 is $C_{12} \cdot h_r$, where h_r is the linearised radiant heat transfer coefficient, around 5·7 W m^{-2} K^{-1} at room temperatures. The radiant flux is $C_{12} \cdot h_r \cdot (T_1 - T_2)$. For the present purpose however, h_r is a constant multiplier in all terms and it is sufficient to work in terms of *geometrical* conductances, (units, m^2).

2.2 *Surface–star point exchange*

It is convenient to consider first a strict delta-to-star transformation of the conductances. We suppose the spherical surface is divided into three areas, so that $A_1 + A_2 + A_3 = S$. A_1 is at a temperature T_1, etc. The conductance linking nodes T_1 and T_2 is $C_{12} = A_1, A_2/S$, etc. Now the external effect of this delta arrangement of conductances is given exactly by the star configuration of conductances K_1, K_2 and K_3, where K_1 links T_1 to the radiant star node T_{rs} etc, provided that K_1 for example is taken as:

$$K_1 = (C_{12} \cdot C_{23} + C_{23} \cdot C_{31} + C_{31} \cdot C_{12})/C_{23}. \quad (5)$$

Upon inserting value for the C's, we find that:

$$K_1 = A_1, \text{etc},$$

that is, the star conductances are simply equal to the patch areas. Clearly the result holds if A_3 for example becomes zero, or if both A_2 and A_3 are zero. This suggests that perhaps K_j might equal A_j, regardless of how many areas the spherical surface is divided into.

To demonstrate that this is indeed so, suppose that the sphere is divided into four discrete areas A_1, A_2, A_3 and A_4, at uniform temperatures $T_1, T_2, \ldots$. We wish first to find the resistance between nodes 1 and 2 (say), when nodes 3 and 4 are adiabatic.

Suppose that a heat flow Q is injected into node 1 and withdrawn from node 2. By continuity at node 1, we have:

$$C_{12} \cdot h_r \cdot (T_1 - T_2) + C_{13} \cdot h_r \cdot (T_1 - T_3) + C_{14} \cdot h_r \cdot (T_1 - T_4) = Q. \quad (6)$$

There are three similar equations which can be written after rearrangement as:

$$\begin{bmatrix} S-A_1 & -A_2 & -A_3 & -A_4 \\ -A_1 & S-A_2 & -A_3 & -A_4 \\ -A_1 & -A_2 & S-A_3 & -A_4 \\ -A_1 & -A_2 & -A_3 & S-A_4 \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} (+Q/h_r)(S/A_1) \\ (-Q/h_r)(S/A_2) \\ 0 \\ 0 \end{bmatrix}. \quad (7)$$

By definition, the net resistance (in geometrical units) between nodes 1 and 2 is:

$$R_{12} = \frac{T_1 - T_2}{Q/h_r}. \quad (8)$$

T_2 can be set equal to zero and so the second row and column of equations (7) can be deleted. After solving for T_1, we find:

$$R_{12} = \frac{1}{A_1} + \frac{1}{A_2}. \quad (9)$$

But this is just the resistance between these nodes in a star based system when the star conductances K_1 and K_2

are A_1 and A_2 respectively. Thus the surface-star network is exactly equivalent in its external effect to the parent surface-surface network. Clearly, this applies to any two nodes when the spherical surface is divided into four portions, and a similar argument shows this to be the case when the sphere is divided into five or more portions.

This result contrasts with the case for an enclosure with plane surfaces. The radiant exchange in an enclosure composed of four or more plane surfaces cannot be represented *exactly* by a star pattern of conductances. The present author has shown [4] that radiant exchange in a rectangular enclosure can in fact be closely modelled by a star based system—for a cubic enclosure the representation is almost exact—but the demonstration involves some computational labour.

3. THE SURFACE CONDUCTANCE β VALUES

In [4] the author defined a quantity β_j for a black body surface of area A_j in a rectangular enclosure where:

$$\beta_j = \frac{\text{conductance from the surface to a large enclosing black body surface}}{\text{conductance from the surface to the enclosure radiant star node}}, \qquad (10)$$

so that the star conductance has a value A_j/β_j. It appeared from tests on a wide variety of enclosure shapes that β_j could be expressed closely as a function of the area concerned in relation to the total enclosure area. We define:

$$f_j = A_j/\Sigma A_j$$

summing over all the enclosure surfaces. Then:

$$\beta_j \simeq 1-f_j-3.53(f_j^2-\tfrac{1}{2}f_j)+5.04(f_j^3-\tfrac{1}{4}f_j), \qquad (11)$$

with a standard deviation of 0.0067 (a small value since β_j values lie between $\frac{1}{2}$ and 1). β_j is mainly estimated as $1-f_j$; the further terms represent only a small deviation from this relation.

β_j values can similarly be found for spherical surfaces. Consider for convenience a patch on a sphere having the form of a circular cap of area A_j whose diameter subtends an angle 2ϕ at the centre of the sphere. Its conductance to the radiant star node is A_j, as shown above. However, as far as radiation is concerned the effective area of emission is not A_j itself, but a reduced area A_j' say—the area in fact of a plane membrane stretched across its perimeter. See [2], p. 66. The radiation to a large enclosing surface is proportional to A_j'. The β_j value for A_j is accordingly:

$$\beta_j = A_j'/A_j. \qquad (12)$$

From the geometry of the sphere, we find:

$$\beta_j = \tfrac{1}{2}(1+\cos\phi). \qquad (13)$$

Similarly:

$$f_j = A_j/S = \tfrac{1}{2}(1-\cos\phi). \qquad (14)$$

Thus for a sphere we have the exact relation:

$$\beta_j = 1-f_j. \qquad (15)$$

This form amounts to being an idealisation of the relation for a rectangular enclosure.

4. THE AVERAGE OBSERVABLE TEMPERATURE

The radiant star node is a convenient fiction and has no physical significance; it is meaningless to input radiation there. However, if a radiant flux Q_r is taken to act there, and the spherical surface is taken to be at zero, its temperature is given by:

$$T_{rs} = Q_r/(S\cdot h_r). \qquad (16)$$

To arrive at a physically meaningful temperature, suppose that the radiant input is supplied by a small source placed at the centre of the sphere, and that its effect is sensed by a small black body spherical probe, of diameter d, distant r from the centre of the sphere. The area of the probe intercepting radiation from the source is $(1/4)\pi d^2$ and the area radiating to the sphere is πd^2. Heat balance at the probe then indicates its temperature to be:

$$T_{rp} = Q_r/(16\pi r^2\cdot h_r). \qquad (17)$$

T_{rp} varies from a high value near the source to a low value at the sphere. Its average value T_{rv} over the volume of the sphere is found as:

$$T_{rv} = \frac{\int T_{rp}\cdot dV}{\int dV}, \qquad (18)$$

integrating over the radius of the sphere. The appropriate element of volume dV is here equal to $4\pi r^2\cdot dr$. We find:

$$T_{rv} = (3/4)\cdot Q_r/(S\cdot h_r). \qquad (19)$$

A comparison of equations (16) and (19) shows that the average observable temperature T_{rv} is 25% *lower* than the value obtained by assuming that the heat is input at the radiant star temperature T_{rs} in a sphere.

The author has examined this point for a rectangular enclosure: for a cubic enclosure, $T_{rv}/T_{rs} = 0.915/0.833 = 1.098$ (see [4], Tables 3 and 5, with some changes in notation), so that T_{rv} is some 10% *higher* than T_{rs} in a cube. There is thus an apparent conflict between the values of T_{rv}/T_{rs} for a sphere and a cube. This is discussed in the following section.

5. COMPARISON OF VALUES FOR SPHERICAL AND CUBIC ENCLOSURES

A study of radiation within a sphere is only of interest insofar as it helps to make clear the case of radiant exchange in a room. Clearly, the rectangular room closest in shape to a sphere is a cube, possibly having the same volume, and in this section we estimate the values of T_{rv} and of T_{rs} for a cube as estimated from the values for a sphere. They can then be compared with the directly computed values for a cube.

5.1. The value of T_{rv}

The value of T_{rv} for a sphere is $0.75 \cdot (Q_r/h_r)$/(area of a sphere). For a cube of side l having the same volume as a sphere of radius R;

$$\frac{\text{area of cube}}{\text{area of sphere}} = 2 \times (3/4\pi)^{1/3} = 1.241.$$

So:

$$T_{rv}(\text{sphere}) = 0.75 \times 1.241\,(Q_r/h_r)/\text{(area of cube of same volume)}$$
$$= 0.931 \times (Q_r/h_r)/\text{(area of cube)}. \quad (20)$$

5.2. The value of T_{rs}

The value of T_{rs} for a sphere is $1.0 \times (Q_r/h_r)$/(area of sphere). Now we have already noted that the radiation from a black body cavity is not proportional to its area but rather to the area of a plane membrane stretched across its mouth. Suppose that from the sphere we slice off six segmental shells with circular peripheries, of such a size that each just touches four of the others, and we replace the circular holes by plane surfaces so as to leave the sphere looking like a partial cube. This leaves the conductances between the areas concerned unaltered, but the area of the partial cube is less than that of the sphere. It is easily shown that:

$$\frac{\text{area of partial cube}}{\text{area of sphere}} = \frac{4\pi R^2 - 6 \times (2-\sqrt{2})\pi R^2 + 6 \times \frac{1}{2}\pi R^2}{4\pi R^2} = 0.871. \quad (21)$$

So:

$$T_{rs} = 1.0 \times 0.871 \times (Q_r/h_r)/\text{(area of partial cube)}.$$

5.3. Comparison of values

The values for T_{rv} and of T_{rs} for a cube are given by the $l/h = d/h = 1$ entries in Tables 5 and 3 of [4] and those for sphere, related as far as possible to a cube-shaped volume are given above. They are listed for comparison below.

T_{rv}(cube) $= 0.915(Q_r/h_r)$/area | T_{rs}(cube) $= 0.833(Q_r/h_r)$/area

T_{rv}(sphere) $= 0.931(Q_r/h_r)$/area | T_{rs}(sphere) $= 0.871(Q_r/h_r)$/area.

Thus the values for T_{rv} are close; there is no reason why they should be exactly equal. The T_{rs} values too are close but again we do not expect them to be the same. In each case, the sphere values exceed the cube values a little.

6. DISCUSSION

It is known that the exchange of radiation between the surfaces of a rectangular enclosure does not depend strongly upon the shape of the enclosure and it might be expected that radiant exchange would not be too much affected by whatever shape within broad limits the enclosure had: the important requirement is that the surfaces form a closed configuration. This note has demonstrated that radiation exchange in the simplest of all closed configurations—a spherical cavity—is very easy to handle and is free from the algebraic and computational complexity of exchange in a rectangular room. Specifically, the view factor relationship for patches on a spherical surface is elementary, and so the expression for the surface to surface conductances. It turns out that the pattern of surface-to-surface conductances transforms exactly to a surface-to-star point configuration, regardless of how many areas the spherical surface is divided into. This eliminates the "least squares" procedure that had to be adopted in [4] to size the surface-to-star point conductances in a rectangular room. Furthermore, it is very easy to compute the average observable radiant temperature in a spherical cavity when the radiant source is placed at the centre; this calculation is very much more involved for a rectangular room.

The other considerations mentioned in the Introduction relating to surface emissivity, air temperature, comfort temperature and the evolution of a room global temperature—rad-air temperature—would be, if pursued, the same for a sphere as for a room. Despite the aritificiality of the sphere as building model, the author feels that it provides an effective way for an initial presentation of the theory of radiant exchange in a room.

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