Building and Environment, Vol. 28, No. 3, pp. 301–309, 1993.
Printed in Great Britain.

0360–1323/93 $6.00+0.00

7492

Computer Generation of Semi-Symbolic Thermal Network Functions of Buildings

T. Y. CHEN*
A. K. ATHIENITIS*

A systematic computer method is presented for generating the symbolic transfer functions of buildings, which have several inherent merits in sensitivity analysis, optimum design and control studies. After introducing the concept of a generalized-node admittance matrix, a new formulation particularly suitable for the semi-symbolic (i.e. some of the network parameters being symbols) network analysis of buildings is described. An algorithm based on both the new formulation and an algebraic method is further developed in order to eliminate as many invalid symbol combinations as possible for the efficient generation of semi-symbolic thermal-network functions of buildings. An example demonstrating the application and efficiency of this method is included.

NOMENCLATURE

Q	heat flow W
T	temperature K
y, Y	thermal admittance W K^{-1}
Z	thermal impedance K W^{-1}
C_p	thermal capacity J K^{-1}
s	Laplace transform variable or symbolic entry
g	transfer conductance W K^{-1}
h	transfer temperature ratio
H	transfer function
A	matrix or node-to-branch incidence matrix
B	loop matrix
C	cutset matrix
S	diagonal matrix with symbolic entries
V	matrix with numerical entries

Subscripts

o	source
j, k, l, m	node numbers
b	branch
n	node
t	tree
s	symbol
v	value
γ_j	index of rows or columns in a matrix
(γ), (i), (j)	a set of the indices of rows or columns in a matrix

INTRODUCTION

THERE ARE two common techniques used in building thermal design and control studies, time domain and frequency domain techniques. The time domain technique can be applied to any systems, including nonlinear and time-varying ones. In many cases, however, nonlinear elements in a network can be linearized and then the whole system becomes linear. For linear systems, the frequency domain technique has several advantages over the time domain technique [5], such as elimination of errors from the discretization of massive walls in the time domain simulation, flexibility in detail of both the simulation and the weather model with a variable number of harmonics [5], easy simulation of large thermal networks [3] and simplification of sensitivity analysis of building thermal parameters [4] by means of network decomposition techniques known as diakoptics. For a complete discussion of diakoptics and its application to building energy analysis, one may refer to [22] and [3, 4]. A potential advantage, which has not been exploited in building thermal design and control studies, is that the symbolic network analysis technique in the frequency domain can be utilized to further facilitate the sensitivity analysis, optimum design and control studies of buildings.

Symbolic network functions have several inherent merits [7, 11]. Because some original design parameters of interest may be retained as symbols in the model, analytical sensitivities on a small scale and optimum design on a large scale could be easier. In addition, when transfer functions can be obtained with s kept as a symbol, their evaluation with different frequencies could be made without the procedures of both solving the simultaneous equations at each harmonic and using fitting techniques for obtaining the transfer function in s-domain [6]. This is particularly useful for frequency response analysis of building thermal processes in the design and tuning of feedback controllers. Moreover, for small building networks with all design parameters in symbols and for large building networks with a few design parameters in symbols, the symbolic analysis technique provides insights into the effect of design parameters on the thermal performance of buildings. Furthermore, there are still such other advantages as control of error in numerical calculation, simplification of time domain calculations by means of inversion of the Laplace transform, sim-

* Centre for Building Studies, Concordia University, 1455 de Maisonneuve Blvd W., Montreal, Quebec, Canada H3G 1M8.

ultaneous evaluation of several network functions and facilitation of statistical analysis as pointed out by Lin [7] and by Singhal *et al.* [11].

Up to now, analytical solutions in the frequency domain for the thermal network of buildings have been deduced by hand. Examples include an analytical model for a five-node network with lumped parameter elements by Kirkpatrick *et al.* [2] and one with a two-node network including a distributed parameter element by Athienitis *et al.* [1]. It is almost impossible to obtain symbolic network functions for a large thermal network without computer aid. A systematic computer method is therefore obviously necessary for this purpose.

Several methods are available in the theory of electric circuits and systems for the symbolic network analysis. Mielke's signal flowgraph formula [8] avoids the invalid calculation of cancelling terms by combining topological network information and is more efficient than the other methods based on Mason's formula. The formulation taking advantages of the node admittance matrix developed by Alderson and Lin [9, 10] has a small number of equations. However, its parameter extraction process is somehow complicated because each symbol appears at four different entries in the coefficient matrix. Singhal and Vlach's numerical interpolation method [11, 12] has been shown computationally competitive with Alderson and Lin's parameter extraction method [12], due to utilising such powerful numerical techniques as FFT algorithms, sparse matrix methods and adjoint-network approaches and eliminating more invalid symbol combinations before calculation. Nevertheless, as commented by Sannuti and Puri [13], Mielke's signal flowgraph formula [8] is only suitable for the network analysis with all the parameters in symbols while Alderson and Lin's parameter extraction and Singhal and Vlach's numerical interpolation are only suitable for the network with a few parameters in symbols.

The formulation presented by Sannuti and Puri [13] seems to be conceptually simple and computationally competitive as compared with the methods mentioned previously. It is suitable for both cases where all or some of the network parameters are represented by symbols. When all the network parameters are symbols, it eliminates more invalid symbol combinations before calculating any determinant than the Mielke's method permits [8]. For semi-symbolic networks, it is simple, direct and in general computationally competitive to the interpolation method, as compared by Sannuti and Puri [13].

Thermal networks of buildings have their own characteristics. In a detailed building thermal network, the number of branches is much more than that of nodes. In addition, there are only several thermal design parameters of buildings that can be chosen by designers while many other parameters in the thermal network of buildings take or approximately take constant values. If Sannuti and Puri's formulation is applied to the thermal design and control of buildings, the coefficient matrix for the thermal network of buildings may be very large due to a great number of branches in the network. This may lead to inconvenient generation of formulation because fundamental cutset and loop matrices are interdependent and considerable effort is required to obtain them [17]. Moreover, their method may be unable to avoid generating some symbol combinations that are topologically invalid. A new formulation is therefore introduced in order to reduce the number of variables. An efficient algorithm based on both the new formulation and the algebraic method presented by Sannuti and Puri [13] is further developed, which may allow one to eliminate more invalid symbol combinations in semi-symbolic network analysis than the method of Sannuti and Puri.

A NEW FORMULATION FOR SEMI-SYMBOLIC THERMAL NETWORKS

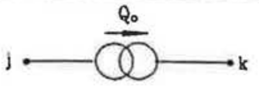
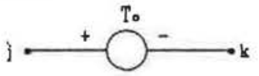
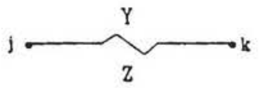
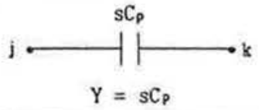
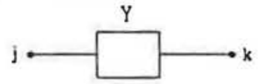
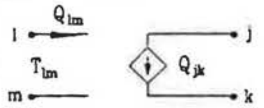
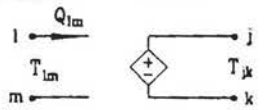
As mentioned previously, the nodal formulation of Alderson and Lin involves a small number of variables, but its parameter extraction process is not straightforward. Sannuti and Puri's formula has only half of the number of variables contained in Mielke's hybrid system of equations, and symbolic analysis based on the former is more efficient. However, Sannuti and Puri's formulation may contain excess topological information for the parameter extraction of semi-symbolic networks in which the number of branches is much more than that of nodes. The advantages of Alderson and Lin's and Sannuti and Puri's formulations could be taken at the same time when the concept of a generalized-node admittance matrix is introduced and the topological information of a given network is appropriately utilized. It is known that a fundamental cutset divides a network into two isolated parts, one of which may be regarded as a generalized node. Utilizing this concept, all of the numerical parameters in the cotree may form a generalized-node admittance submatrix in order to reduce the size of coefficient matrix for the thermal network of buildings.

In the thermal network of buildings, heat flow sources, such as solar radiation, may be equivalently transformed into dependent heat flow sources controlled by temperature variables. Hence, thermal networks in which there are only dependent heat flow and temperature sources controlled by temperature variables besides thermal impedance, capacitance and independent source elements are considered here.

The new formulation, like the others, is based on the two thermal balance laws, which are analogous to the Kirchhoff's Current Law and the Kirchhoff's Voltage Law, and the element constitutive relations in the frequency domain, which are shown in Table 1. It should be noted from Table 1 that the thermal admittance here is defined to be $Y = Q_{jk}/T_{jk}$, which is analogous to the electric admittance. For instance, when T_{jk} denotes swing in indoor air temperature and Q_{jk} is the rate of heat flow through the internal surface of the construction, Y is called as the self-admittance in this paper, which corresponds to the thermal admittance in [24] and the CIBSE Guide [25].

For a given thermal network with n nodes, a complete tree should first be selected in such a way that edges with controlling and controlled temperature variables and as many edges with symbolic parameters as possible act as tree branches while edges with controlled heat flow sources do not. The temperature potentials of tree branches, the necessary heat flows of several cotree branches with symbols, and node temperatures are chosen as a set of basic variables. Providing that any edge with

Table 1. Basic elements in the thermal network

ELEMENTS	SYMBOLS	CONSTITUTIVE EQUATIONS
HEAT FLOW SOURCE		$Q_{jk} = Q_o$
TEMPERATURE SOURCE		$T_{jk} = T_o$
LUMPED ADMITTANCE OR IMPEDANCE		$YT_{jk} - Q_{jk} = 0$ or $T_{jk} - ZQ_{jk} = 0$
LUMPED THERMAL CAPACITY		$sC_p T_{jk} - Q_{jk} = 0$
GENERAL ADMITTANCE		$YT_{jk} - Q_{jk} = 0$
TEMPERATURE-CONTROLLED HEAT FLOW SOURCE		$Q_{lm} = 0$ $Q_{jk} = gT_{lm}$
TEMPERATURE-CONTROLLED TEMPERATURE SOURCE		$Q_{lm} = 0$ $T_{jk} = hT_{lm}$

a controlled temperature source in a tree does not form a fundamental cutset, the law of conservation of thermal energy can be expressed by

$$\mathbf{YT_b + CQ_b + Y_nT_n = 0} \tag{1}$$

where $\mathbf{T_b}$ represents a tree-branch-temperature vector corresponding to the passive tree branches; $\mathbf{Q_b}$ is a heat flow vector corresponding to the passive cotree branches with symbolic parameters; $\mathbf{T_n}$ stands for an $n-1$ dimensional node-temperature vector; $\mathbf{C}$ denotes an appropriate cutset matrix; $\mathbf{Y}$ betokens a diagonal admittance matrix; $\mathbf{Y_n}$ is a generalized-node admittance matrix. The rules for forming it are as follows:

Assume that a tree branch intersected by a fundamental cutset i is directed away from the generalized node i and that node j belongs to the generalized node i but node k does not.

(a) For an admittance element y connected between nodes j and k, $+y$ appears at the entry $Y_n(\mathrm{i, j})$ while $-y$ at $Y_n(\mathrm{i, k})$.

(b) For a dependent heat flow element g leaving from node j to k controlled by temperature difference T_{lm} between nodes l and m, $+g$ appears at both $Y_n(\mathrm{j, l})$ and $Y_n(\mathrm{k, m})$ while $-g$ at $Y_n(\mathrm{j, m})$ and $Y_n(\mathrm{k, l})$; for the heat flow leaving from node k to j, the sign of the entries is opposite.

(c) All the numerical parameters of cotree edges should appear in the generalized-node admittance matrix.

Applying the Kirchhoff's Voltage Law to those cotree edges with symbolic parameters, we obtain

$$\mathbf{BT_b + ZQ_b = 0} \tag{2}$$

where $\mathbf{Z}$ is a symbolic diagonal impedance matrix and $\mathbf{B}$ is an appropriate loop matrix. According to the definition, the transfer function H is the ratio of a desired output x to an independent source x_o.

$$H = \frac{x}{x_o}$$

or

$$-x + Hx_o = 0. \tag{3}$$

In the thermal network of buildings, x may be an indoor air temperature or operative temperature and x_o may be outdoor air temperature, solar radiation or auxiliary heat. For instance, an auxiliary heat source Q may first be replaced with a dependent heat source controlled by indoor temperature T, which can be expressed by $Q = T/H$. Then, it may be rearranged as in the form of Equation (3). The constitutive equation for the element of temperature-controlled heat flow source in Table 1 may be written in the form

$$-\mathbf{T}_b + \mathbf{Z}\mathbf{Q}_b = 0 \tag{4}$$

where $\mathbf{Z}$ is equal to $\mathbf{G}^{-1}$; $\mathbf{G}$ is a diagonal matrix consisting of g_i; g_i is the ratio of Q_{jk}/T_{lm}. The determination of tree-branch-temperature vector $\mathbf{T}_b$ in terms of node-temperature vector in Equation 1 is given by

$$\mathbf{T}_b - \mathbf{A}_t^T\mathbf{T}_n = 0 \tag{5}$$

where $\mathbf{A}_t^T$ is the transpose of an appropriate node-to-tree-branch incidence matrix.

When Equations (1) through (5) are put into one matrix equation and properly partitioned, we have a new formulation in the form

$$\begin{bmatrix} \mathbf{Y1} & 0 & 0 & 0 & \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{Y}_{1n} \\ 0 & \mathbf{Y2} & 0 & 0 & \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{Y}_{2n} \\ 0 & 0 & \mathbf{Y3} & 0 & \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{Y}_{3n} \\ 0 & -\mathbf{I} & 0 & \mathbf{H4} & 0 & 0 & 0 \\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 & \mathbf{Z5} & 0 & 0 \\ 0 & 0 & -\mathbf{I} & 0 & 0 & \mathbf{Z6} & 0 \\ \mathbf{I}_1 & \mathbf{I}_2 & \mathbf{I}_3 & \mathbf{I}_4 & 0 & 0 & -\mathbf{A}_t^T \end{bmatrix} \begin{bmatrix} \mathbf{T1}_b \\ \mathbf{T2}_b \\ \mathbf{T3}_b \\ \mathbf{T4}_b \\ \mathbf{Q5}_b \\ \mathbf{Q6}_b \\ \mathbf{T}_n \end{bmatrix} = 0 \tag{6}$$

where $\mathbf{T1}_b$, $\mathbf{T2}_b$ and $\mathbf{T3}_b$, and $\mathbf{T4}_b$ form a complete-tree-branch-temperature subvector $\mathbf{T}_b$, corresponding to uncontrolling, controlling and controlled edges, respectively; $\mathbf{Q5}_b$ and $\mathbf{Q6}_b$ constitute a heat flow subvector $\mathbf{Q}_b$, corresponding to uncontrolling and controlled edges, respectively; $\mathbf{Y1}$, $\mathbf{Y2}$ and $\mathbf{Y3}$, and $\mathbf{Z5}$ and $\mathbf{Z6}$ betoken partitioned diagonal admittance and impedance submatrices forming $\mathbf{Y}$ and $\mathbf{Z}$, respectively; $\mathbf{H4}$ is a diagonal submatrix of nondimensional parameters. $\mathbf{Y}_{1n}$, $\mathbf{Y}_{2n}$ and $\mathbf{Y}_{3n}$ constitute a generalized-node admittance submatrix $\mathbf{Y}_n$ in Equation (1); $\mathbf{B}_{ij}$ and $\mathbf{C}_{ij}$ represent appropriately partitioned loop and cutset submatrices forming $\mathbf{B}$ and $\mathbf{C}$, respectively; $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{I}_4$ together form an n-1 × n-1 dimensional identity submatrix; $\mathbf{I}$ is an identity submatrix of appropriate order.

It should be noted that the determinant of the coefficient matrix in Equation (6) should be equal to zero, otherwise the homogeneous equations could only have a trivial solution. Hence, the transfer function can be obtained by the sorting approach [7], in which all the terms with a transfer function symbol are sorted out for the denominator, multiplied by -1. The numerator of the transfer function is equal to the sum of the other terms.

AN ALGORITHM BASED ON BOTH THE NEW FORMULATION AND SANNUTI AND PURI'S ALGEBRAIC METHOD

Let us consider all tree branches with symbolic parameters first. The coefficient matrix may be partitioned into four blocks

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{7}$$

where

$$\mathbf{A}_{11} = \begin{bmatrix} \mathbf{Y1} & 0 & 0 & 0 & \mathbf{C}_{11} & \mathbf{C}_{12} \\ 0 & \mathbf{Y2} & 0 & 0 & \mathbf{C}_{21} & \mathbf{C}_{22} \\ 0 & 0 & \mathbf{Y3} & 0 & \mathbf{C}_{31} & \mathbf{C}_{32} \\ 0 & -\mathbf{I} & 0 & \mathbf{H4} & 0 & 0 \\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 & \mathbf{Z5} & 0 \\ 0 & 0 & -\mathbf{I} & 0 & 0 & \mathbf{Z6} \end{bmatrix}$$

$$\mathbf{A}_{12} = [\mathbf{Y}_n \quad 0]^T, \quad \mathbf{Y}_n = [\mathbf{Y}_{1n} \quad \mathbf{Y}_{2n} \quad \mathbf{Y}_{3n}]$$

$$\mathbf{A}_{21} = [\mathbf{I} \quad 0], \quad \mathbf{I} = [\mathbf{I}_1 \quad \mathbf{I}_2 \quad \mathbf{I}_3 \quad \mathbf{I}_4]$$

$$\mathbf{A}_{22} = -\mathbf{A}_t^T.$$

It can be observed that the submatrices $\mathbf{A}_{12}$, $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ consist of all their entries with numerical values while the submatrix $\mathbf{A}_{11}$ has all its principal diagonal entries with symbolic parameters. Applying the generalized algorithm of Gauss [15], the determinant Δ of the coefficient matrix may be obtained by

$$\Delta = |\mathbf{A}_{11} - \mathbf{A}_{12} \quad \mathbf{A}_{22}^{-1} \quad \mathbf{A}_{21}||\mathbf{A}_{22}|. \tag{8}$$

Substituting (7) into (8), we obtain

$$\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} = -\begin{bmatrix} \mathbf{Y}_n(\mathbf{A}_t^T)^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{9}$$

Partition the above matrix into such blocks that it matches with the partitioned matrix $\mathbf{A}_{11}$.

$$\mathbf{Y}_n(\mathbf{A}_t^T)^{-1} = \begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} & \mathbf{Y}_{13} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} & \mathbf{Y}_{23} \\ \mathbf{Y}_{31} & \mathbf{Y}_{32} & \mathbf{Y}_{33} \end{bmatrix}. \tag{10}$$

Substituting (8), (9) and (10) into (7), we have

$$\Delta = |\mathbf{S}| + |\mathbf{V}| \tag{11}$$

where

$$\mathbf{S} = \mathrm{diag}\,[\mathbf{Y1} \quad \mathbf{Y2} \quad \mathbf{Y3} \quad \mathbf{H4} \quad \mathbf{Z5} \quad \mathbf{Z6}]$$

$$\mathbf{V} = \left[\begin{array}{cccc|cc} \mathbf{Y}_{11} & \mathbf{Y}_{12} & \mathbf{Y}_{13} & 0 & \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} & \mathbf{Y}_{23} & 0 & \mathbf{C}_{21} & \mathbf{C}_{22} \\ \mathbf{Y}_{31} & \mathbf{Y}_{32} & \mathbf{Y}_{33} & 0 & \mathbf{C}_{31} & \mathbf{C}_{32} \\ \hline 0 & -\mathbf{I} & 0 & 0 & 0 & 0 \\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 & 0 & 0 \\ 0 & 0 & -\mathbf{I} & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \hline \mathbf{V}_{21} & \mathbf{V}_{22} \end{array}\right].$$

In Equation (11), $\mathbf{S}$ is a diagonal matrix with symbolic entries and $\mathbf{V}$ is a dimensional matrix with all its entries numerical. Using Cayley's expansion of a determinant [13, 18], Δ may be obtained by

$$\Delta = \sum_{j=0}^{k} \sum_{\Gamma} \left[\prod_{i=1}^{j} s_{\gamma_i}\right] |\mathbf{V}_{(\gamma)}| \tag{12}$$

where

$$|\mathbf{V}_{(\gamma)}| = 1, \quad \text{when } j = k;$$

$$\prod_{i=1}^{j} s_{\gamma_i} = 1, \quad \text{when } j = 0.$$

(γ) is a set consisting of $\gamma_1, \gamma_2, \ldots, \gamma_j$; s denotes an entry in the diagonal matrix $\mathbf{S}$; γ_i represents the row and column index of the entry s; Γ is the set of all possible (γ). $\mathbf{V}_{(\gamma)}$ is a submatrix taken from the matrix $\mathbf{V}$ in Equation (11) by deleting the rows and columns corresponding to the set (γ), the determinant of which is the coefficient of symbol combinations.

In expanding the determinant, some symbol combinations may be invalid because the minor $|\mathbf{V}_{(\gamma)}|$ may be equal to zero. When using Cayley's expansion, therefore, the theorem (Appendix B) given by Sannuti and Puri [13] may first be used in order to eliminate those invalid terms before calculating any minor. Close examination of Equation (11) shows that the submatrix $\mathbf{V}_{22}$ is composed of all its zero entries and that $\mathbf{V}_{12}$ and $\mathbf{V}_{21}$ contain the useful topological information of a given network. Hence, more invalid terms may be eliminated if some conditions stricter than those (Appendix B) given by Sannuti and Puri [13] are introduced. It is assumed that each of 6 submatrices in the diagonal matrix S in Equation (11) has n_k symbols, m_k represents the number of symbols extracted from each submatrix and then the number of remaining symbols is $n_k - m_k = l_k$. The coefficient matrix $\mathbf{V}_{(\gamma)}$ may be written in the form

$$\mathbf{V}_{(\gamma)} = \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{0} \end{bmatrix}$$

where $\mathbf{U}_{ij}$ is a submatrix taken from the submatrix $\mathbf{V}_{ij}$ in Equation (11) by deleting the rows and columns corresponding to the set (γ); $\mathbf{U}_{12}$ and $\mathbf{U}_{21}$ are $(l_1+l_2+l_3) \times (l_5+l_6)$ and $(l_4+l_5+l_6) \times (l_1+l_2+l_3+l_4)$ submatrices, respectively. It is supposed that there are only n_r rows and n_c columns with nonzero entries in $\mathbf{U}_{12}$ and $\mathbf{U}_{21}$, separately. Then, we have the following inequalities for eliminating invalid symbol combinations.

The constraint conditions of inequalities: The determinant of $\mathbf{V}_{(\gamma)}$ could be nonzero only if $n_r \geqslant l_5+l_6$ and $n_c \geqslant l_4+l_5+l_6$.

Proof. Expanding the determinant of $\mathbf{V}_{(\gamma)}$ from the last l_5+l_6 columns according to Laplace's theorem (Appendix A), if $n_r < l_5+l_6$, the determinant of $\mathbf{V}_{(\gamma)}$ will be equal to zero because there are, at least, $(l_5+l_6)-n_r$ rows with all zero entries in any minor of order l_5+l_6 taken from the last l_5+l_6 columns. Similarly, we can prove the other condition of $n_c \geqslant l_4+l_5+l_6$.

Let us now consider thermal networks in which some parameters of tree branches are expressed by numerical values. Although the principle of calculation to be presented can be applied to any thermal network, to simplify notations, it is assumed that only some of the parameters of uncontrolling edges in the tree are numerical values. Then, Equation (11) may be rearranged and repartitioned in the form

$$\mathbf{S} = \mathrm{diag}\,[\mathbf{0} \quad \mathbf{Y1}_s \quad \mathbf{Y2} \quad \mathbf{Y3} \quad \mathbf{H4} \quad \mathbf{Z5} \quad \mathbf{Z6}]$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix} \tag{13}$$

where

$$\mathbf{V}_{11} = [\mathbf{Y}_v + \mathbf{Y1}_v]$$

$$\mathbf{V}_{12} = [\mathbf{Y}_{v1} \quad \mathbf{Y}_{v2} \quad \mathbf{Y}_{v3} \quad \mathbf{0} \quad \mathbf{C}_{v1} \quad \mathbf{C}_{v2}]$$

$$\mathbf{V}_{21} = [\mathbf{Y}_{1v} \quad \mathbf{Y}_{2v} \quad \mathbf{Y}_{3v} \quad \mathbf{0} \quad \mathbf{B}_v \quad \mathbf{0}]^{\mathrm{T}}$$

$$\mathbf{V}_{22} = \left[\begin{array}{cccc:cc} \mathbf{Y}_s & \mathbf{Y}_{s2} & \mathbf{Y}_{s3} & \mathbf{0} & \mathbf{C}_{s1} & \mathbf{C}_{s2} \\ \mathbf{Y}_{2s} & \mathbf{Y}_{22} & \mathbf{Y}_{23} & \mathbf{0} & \mathbf{C}_{21} & \mathbf{C}_{22} \\ \mathbf{Y}_{3s} & \mathbf{Y}_{32} & \mathbf{Y}_{33} & \mathbf{0} & \mathbf{C}_{31} & \mathbf{C}_{32} \\ \hdashline \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_s & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] = \left[\begin{array}{c:c} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \hdashline \mathbf{W}_{21} & \mathbf{W}_{22} \end{array}\right]$$

where $\mathbf{Y1}_v$ and $\mathbf{Y1}_s$ denote the numerical and symbolic parameters of uncontrolling tree branches, respectively. The other submatrices with the subscripts v and s are partitioned, according to $\mathbf{Y1}_v$ and $\mathbf{Y1}_s$.

There may be two approaches that can be employed to find the determinant of the above matrix.

Approach 1. The algorithm for thermal networks in which all the parameters of tree branches are symbolic may be directly employed. The advantage of this approach is that it can eliminate as many invalid terms as possible before calculating determinants. Nevertheless, the disadvantage is that higher order determinants need to be calculated in the symbol extraction process.

Approach 2. Similar to the approach used before, the generalized algorithm of Gauss may first be employed to reduce the order of the coefficient matrix and the symbol extraction process is then carried out. The advantage of this approach is that lower order determinants-are calculated in the process. However, the disadvantage is that the useful topological information in the coefficient matrix formed properly could be neglected after secondly performing the generalized algorithm of Gauss.

The two approaches can be combined together because whatever kind of methods is applied, the solution should be unique. Therefore, the former may be used to weed out invalid terms while the latter to calculate determinants in the symbol extraction process. It should be noted that this principle of calculation can also be applied to Sannuti and Puri's method for the efficient generation of semi-symbolic network functions.

EXAMPLE APPLICATION

Although the method developed can deal with any detailed model of buildings, as an example, a simplified model may be helpful for both describing and understanding it. The dimensions of a room under consideration are $2.82 \times 2.22 \times 2.24$ m and the window area is $1.17\ \mathrm{m}^2$. The room model with node positions is schematically shown in Fig. 1(a) and its thermal network in Fig. 1(b), which is a five-node model, including node 5 for an outdoor temperature source. Node 1 denotes the interior surface of window glazing, the thermal mass of which is assumed to be negligible. Node 2 represents the interior surface of ceiling, the thickness of which is so thin that its thermal mass can be treated as a lumped capacity. Node 3 is the interior surface of surrounding walls made up of an inner lining of storage mass material and outer massless insulation. Node 4 betokens indoor air temperature and node 6 is for reference. Surrounding

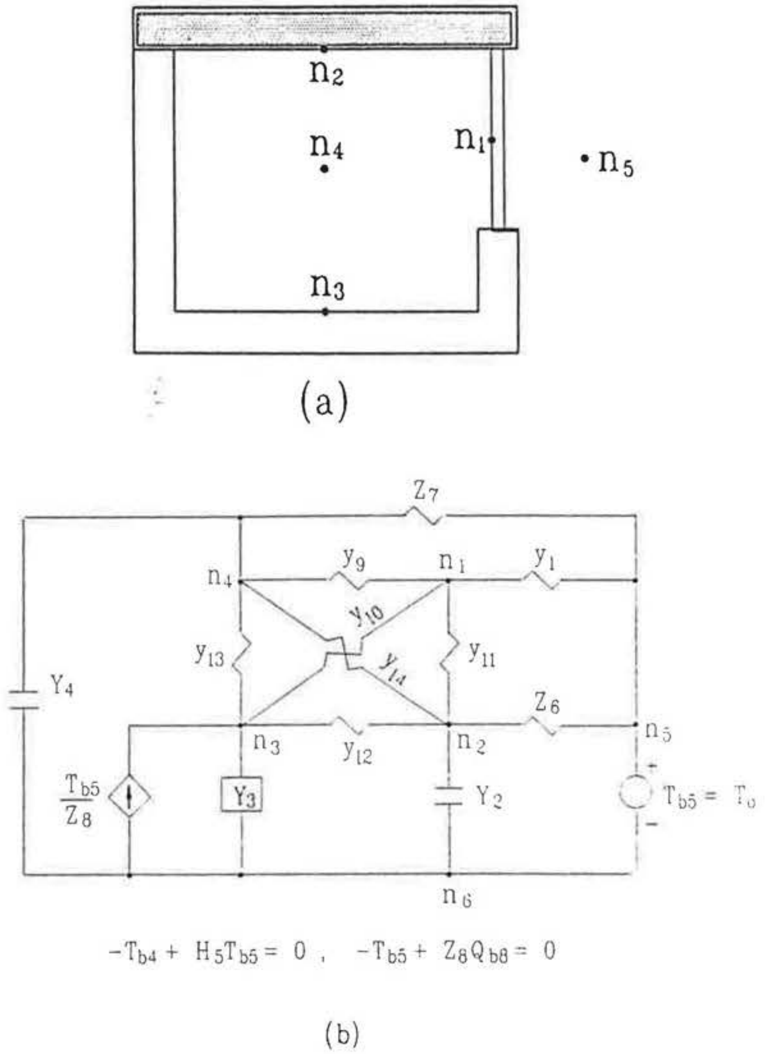


Fig. 1. Thermal network of room.

multilayered walls are modelled by transfer- and self-admittances [1], $1/Z_8$ and Y_3. The heat flow Q_8 through surrounding walls into the room forced by outdoor temperature T_o can be expressed by $Q_8 = T_{b5}/Z_8$ (or Y_8T_o) according to Norton's theorem [1], where Y_8 is the transfer-admittance of surrounding walls and T_{b5} denotes T_o for unified notation. When T_{b5} is treated as a variable, the heat flow is equivalently transformed to a temperature-controlled heat flow source. The capital letters denote symbolic parameters, which are defined as follows: Y_2 is the heat capacity admittance of ceiling; Y_3 the self-admittance of exterior walls; Y_4 the heat capacity admittance of indoor air; H_5 the ratio of indoor temperature T_{in} (or T_{b4}) to outdoor temperature T_o (or T_{b5}); Z_6 and Z_7 the impedances of roof and air infiltration; Z_8 the transfer-impedance of exterior walls. The small letters betoken numerical values, which are as follows: $y_1 = 5.476$ W/K; $y_9 = 3.593$ W/K; $y_{10} = 4.311$ W/K; $y_{11} = 1.655$ W/K; $y_{12} = 29.802$ W/K; $y_{13} = 85.233$ W/K and $y_{14} = 19.282$ W/K.

The steps to obtain the transfer function of a given thermal network are as follows:

Step 1. An independent source, such as outdoor temperature and auxiliary heat, is replaced by a dependent source controlled by the output variable of interest. In this example, the outdoor temperature T_{b5} is first replaced with a dependent temperature source controlled by indoor temperature T_{b4}, which can be expressed by $-T_{b4}+H_5T_{b5} = 0$, according to Equation (3).

Step 2. If necessary, the thermal network of a building, like Fig. 1, and its corresponding graphs of temperature and heat flow, like Fig. 2, may be drawn out and then a complete tree must be selected. Here, the branches from 1 to 5, bold lines in Fig. 2, are chosen as the tree. Note that only some edges, such as tree branches and cotree branches with symbolic parameters, should be oriented in the graph and that the tree branch with the dependent temperature source controlled by the temperature variable does not necessarily need to form a fundamental cutset. For instance, branch 5, a dotted line in the graph b of Fig. 2, is such a tree branch.

Step 3. Following the formula (6), the system of thermal balance equations can be established in the form of Equation (7). In the example, the major submatrices are written as follows

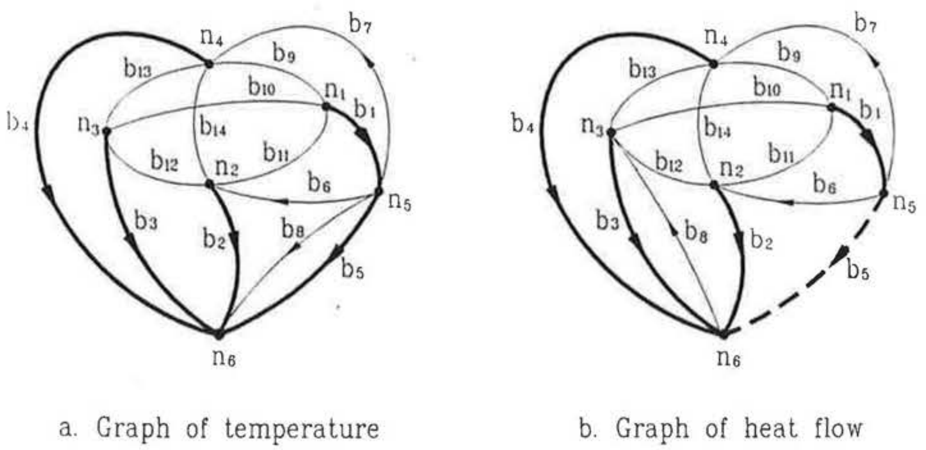


Fig. 2. Graph corresponding to the thermal network in Fig. 1.

$$\mathbf{A}_{11} = \begin{bmatrix} y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & Y_3 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & Y_4 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & H_5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & Z_6 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & Z_7 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & Z_8 \end{bmatrix}$$

$$\mathbf{Y}_n = \begin{bmatrix} y_9+y_{10}+y_{11} & -y_{11} & -y_{10} & -y_9 & 0 \\ -y_{11} & y_{11}+y_{12}+y_{14} & -y_{12} & -y_{14} & 0 \\ -y_{10} & -y_{12} & y_{10}+y_{12}+y_{13} & -y_{13} & 0 \\ -y_9 & -y_{14} & -y_{13} & y_9+y_{13}+y_{14} & 0 \end{bmatrix}$$

$$-\mathbf{A}_t^{\mathrm{T}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Step 4. Because $\mathbf{A}_t^{\mathrm{T}}$ is nonsingular, Equations (8) through (10) can be used to obtain the formula of a determinant of lower order in the form of Equation (11).

Step 5. Search the submatrices $\mathbf{V}_{12}$ and $\mathbf{V}_{21}$ in Equation (11) for nonzero entries and record the topological information, i.e. the row and column indices of those entries with nonzero values.

Step 6. If, like this example, some parameters of elements in the tree take numerical values, the generalized algorithm of Gauss is secondly applied to reduce the order of the coefficient matrix, otherwise skip this step.

Step 7. According to the inequalities presented in the third section, invalid cancelling terms can be eliminated before calculating any determinant and if a symbol combination is valid, Equation (12) is used to find its coefficient.

Step 8. Having sorted out the terms with the distinct symbol H_5, we obtain the transfer function whose denominator is equal to the sum of these terms, multiplied by -1, and numerator to the sum of the rest [7].

The transfer function of T_{b4}/T_{b5} for the thermal network in Fig. 1 is shown in Table 2 after the eight step calculations.

For comparison, Sannuti and Puri's method has also been applied to this example. The result shows that 28 invalid symbol combinations have to be generated before weeding them out while there is no invalid symbol combination among those terms generated by the inequalities given in the third section.

DISCUSSION

As mentioned previously, the new method, like all the other methods for building energy analysis in the frequency domain, is based on the linearization of building systems. This assumption has been proved to be acceptable by Haghighat and Athienitis [5]. They compared and validated the program BEEP [6], which uses frequency domain techniques, with TARP (Thermal Analysis Research Program) and with experimental data. BEEP produces the numerical transfer functions of buildings while the new method generates the symbolic ones. The two computer programs have also been compared with each other with the case described in the example. BEEP gives a more accurate result than the new program because it uses a more detailed building model (eight-node model). A computer program for generating the symbolic transfer function of more detailed models is now under development.

Although the thermal network in Fig. 1 only involves indoor air temperature, the new method may also be used to obtain operative temperature or resultant temperature. One may refer to [23] for a complete procedure for calculating operative temperature by means of network techniques.

Symbolic transfer functions in the frequency domain, like H_5 in Table 2, can be widely applied to building energy analysis and control studies. When all the components of a system consisting of building, HVAC and control subsystems are represented by the Laplace trans-

Table 2. Symbolic Transfer Function $H_5 = N(s)/D(s)$

No.	NUMERATOR $N(s)$ Coefficient	NUMERATOR $N(s)$ Symbolic Term	DENOMINATOR $D(s)$ Coefficient	DENOMINATOR $D(s)$ Symbolic Term
1	-1.34×10^{-5}		1775.89	Z_8
2	1775.89	Z_8	15.04	Y_3Z_8
3	-7.07×10^{-4}	Z_6	76001.16	Z_6Z_8
4	1297.04	Z_7	78575.95	Z_7Z_8
5	5.64×10^{-7}	Y_2Z_6	1775.89	$Y_2Z_6Z_8$
6	15.04	Y_3Z_8	760.17	$Y_3Z_6Z_8$
7	76001.15	Z_6Z_8	1612.59	$Y_3Z_7Z_8$
8	78575.95	Z_7Z_8	1775.89	$Y_4Z_7Z_8$
9	74532.95	Z_6Z_7	261235.30	$Z_6Z_7Z_8$
10	1775.89	$Y_2Z_6Z_8$	15.04	$Y_2Y_3Z_6Z_8$
11	1297.04	$Y_2Z_6Z_7$	78575.95	$Y_2Z_6Z_7Z_8$
12	760.17	$Y_3Z_6Z_8$	15.04	$Y_3Y_4Z_7Z_8$
13	315.54	$Y_3Z_7Z_8$	75706.10	$Y_3Z_6Z_7Z_8$
14	261235.90	$Z_6Z_7Z_8$	76001.16	$Y_4Z_6Z_7Z_8$
15	15.04	$Y_2Y_3Z_6Z_8$	1612.59	$Y_2Y_3Z_6Z_7Z_8$
16	4360.48	$Y_2Z_6Z_7Z_8$	1775.89	$Y_2Y_4Z_6Z_7Z_8$
17	1173.07	$Y_3Z_6Z_7Z_8$	760.17	$Y_3Y_4Z_6Z_7Z_8$
18	19.67	$Y_2Y_3Z_6Z_7Z_8$	15.04	$Y_2Y_3Y_4Z_6Z_7Z_8$

fer function, the frequency response analysis is readily performed. For instance, the Nyquist plots may be drawn out to study the frequency response characteristics of the dynamic control system of buildings and to design feedback controllers [6]. In addition, when the Laplace transform variable s is set equal to $j\omega$, where $j = (-1)^{1/2}$ and ω is the frequency, the simulation of room air temperature or energy consumption in the time domain can be easily carried out by means of superposition of the individual harmonic components [1]. Moreover, the equivalent z-transfer functions for the digital control of dynamic building systems can be obtained through one of the approaches, such as pole-zero mapping and hold equivalence [21]. Furthermore, the time domain response of buildings can also be found from the frequency domain transfer function by means of the numerical Laplace transform inversion [6, 17].

Several efficient procedures may be incorporated in the computer program for automatic formulation and solution of given building models. First, the thermal network of buildings can be identified by the computer program only according to the input information, such as the number of interior surfaces of a room, symbolic and numerical parameters. Second, such heat sources as solar radiation may be automatically modeled by inputting the fractions projected on each of the interior surfaces of the room. Third, a computer procedure given in reference [20] for automatically choosing a complete tree of the network may be adopted with minor modification. Finally, the regulations described in the second section for selecting tree and cotree branches should also be included so that the system equations (6) can be appropriately generated by computer. When all of the above procedures have been combined with the symbolic extraction program, it needs little effort and knowledge of the network theory to carry out the thermal design and control studies of buildings with the symbolic network analysis technique.

CONCLUSION

A systematic method for the symbolic transfer function of buildings has been developed. Because thermal networks of buildings have their own characteristics, a new formulation is presented by introducing the concept of a generalized-node admittance matrix and utilizing the topological information of a given network appropriately. It can greatly reduce the size of the coefficient matrix for the detailed thermal network of buildings as compared with any existing hybrid system of equations. An efficient algorithm based on both the new formulation and Sannuti and Puri's algebraic method is further developed. The example shows that all of invalid symbol combinations of a given thermal network are eliminated before calculation. In a general case, it can be predicted that the new method may eliminate more cancelling terms than the others. This will lead to more efficient generation of semi-symbolic transfer functions of buildings.

REFERENCES

1. A. K. Athienitis, H. F. Sullivan and K. G. T. Hollands, Analytical method, sensitivity analysis, and algorithm for temperature swings in direct gain rooms. *Sol. Energy* **36**, 303–312 (1986).
2. A. T. Kirkpatrick and C. B. Winn, Spectral analysis of the effective temperature in passive solar buildings. *J. Sol. Energy Engng* **106**, 112–119 (1984).
3. A. K. Athienitis, M. Chandrashekar and H. F. Sullivan, Modelling and analysis of thermal networks through subnetworks for multizone passive solar buildings. *J. Appl. Math. Modelling* **9**, 109–116 (1985).
4. A. K. Athienitis, A computer method for systematic sensitivity analysis of building thermal networks. *Bldg. Envir.* **24**, 163–168 (1989).
5. F. Haghighat and A. K. Athienitis, Comparison between time domain and frequency domain computer program for building energy analysis. *Computer-Aided Design* **20**, 525–532 (1988).

6. A. K. Athienitis, M. Stylianou and J. Shou, A methodology for building thermal dynamics studies and control applications. *ASHRAE Trans.* **2**, 839–848 (1990).
7. P. M. Lin, A survey of applications of symbolic network functions. *IEEE Trans. Circuit Theory* **CT-20**, 732–737 (1973).
8. R. R. Mielke, A new signal flowgraph formulation of symbolic network functions. *IEEE Trans. Circuits Syst.* **CAS-25**, 334–340 (1978).
9. G. E. Alderson and P. M. Lin, Integrating Topological and Numerical Method for Semi-Symbolic Network Analysis. *Proc. 8th Allerton Conf. Circuit Theory* 646–654 (1970).
10. G. E. Alderson and P. M. Lin, Computer generation of symbolic network functions—a new theory and implementation. *IEEE Trans. Circuit Theory* **CT-20**, 48–56 (1973).
11. K. Singhal and J. Vlach, Generation of immittance functions in symbolic form for lumped distributed active networks. *IEEE Trans. Circuits Syst.* **CAS-21**, 57–67 (1974).
12. K. Singhal and J. Vlach, Symbolic analysis of analog and digital circuits. *IEEE Trans. Circuits Syst.* **CAS-24**, 598–609 (1977).
13. P. Sannuti and N. N. Puri, Symbolic network analysis—an algebraic formulation. *IEEE Trans. Circuits Syst.* **CAS-27**, 679–687 (1980).
14. H. W. Turnbull, *The Theory of Determinants, Matrices and Invariants*. Dover Pub., New York (1960).
15. F. R. Gantmacher, *The Theory of Matrices*. Chelsea Pub., New York (1959).
16. W. M. Rohsenow, J. P. Hartnett and E. N. Ganic, *Handbook of Heat Transfer*. McGraw-Hill (1985).
17. J. Vlach and K. Singhal, *Computer Methods for Circuit Analysis and Design*. Van Nostrand Reinhold, New York (1983).
18. F. E. Hohn, *Elementary Matrix Algebra*. Macmillan, New York (1964).
19. W. K. Chen, *Linear Network and Systems*. Wadsnorth, Belmont (1983).
20. O. Wing, *Circuit Theory with Computer Methods*. McGraw-Hill (1978).
21. G. F. Franklin and J. D. Powell, *Digital Control of Dynamic Systems*. Addison-Wesley (1980).
22. A. Brameller, M. N. John and M. R. Scott, *Practical Diakoptics for Electrical Networks*. Chapman and Hall, London (1969).
23. A. K. Athienitis and J. G. Shou, Control of radiant heating based on the operative temperature. *ASHRAE Trans.* **97**, Part 2, (1991).
24. M. G. Davies, The thermal admittance of layered walls. *Bldg. Sci.* **8**, 207–220 (1973).
25. *CIBSE Guide*. Staples Printers St Albans, London (1986).

APPENDIX A

Theorem 1 (Laplace's theorem of a determinant) [17]: Consider a square matrix of order n. If any k ($1 \leqslant k \leqslant n-1$) rows of **A** are selected, each of all possible k-order minors taken from these rows is multiplied by its cofactor and then the sum of these products is equal to the determinant of the matrix **A**.

APPENDIX B

Theorem 2 [13]: Consider a partitioned matrix A of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{1(j)} & \mathbf{A}_2 \\ \mathbf{I}_{(i)(j)} & 0 \end{bmatrix}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are $l \times m$ and $l \times m_2$ submatrices, respectively; **I** is an $m \times m$ identity submatrix; **(i)** is a set of the indices of l_1 rows deleted from the submatrix **I**; **(j)** is a set of the indices of m_1 columns deleted from the submatrices $\mathbf{A}_1$ and **I**. Then, the determinant of **A** could be nonzero only if $m-l_1 \leqslant m-m_1$ and if (j) is either a subset of (i) or the same as (i). Under this condition, the determinant of **A** is obtained by

$$\Delta = (-1)^p |\mathbf{A}_{1(j+k)} \mathbf{A}_2|$$

where

$$p = l(m-l_1) + \sum_{n=1}^{m-l_1} (k_n + n);$$

(j+k) denotes a set of the indices of columns deleted from $\mathbf{A}_1$ and the sum of the two sets (j) and **(k)**; **(k)** is a set consisting of k_n (n is from 1 to $m-l_1$) and the complement of the set **(i)** in the submatrix **I**.